

ON INFINITE ORTHOGONAL MATRICES

BY

MONROE HARNISH MARTIN

A Dissertation

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS
UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

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INTRODUCTION.

If one interprets the elements m_{jk} ($j, k = 1, 2, \dots, n < +\infty$) of an arbitrary real or complex n rowed square matrix $\mathbf{M} = \| m_{jk} \|$ as the coördinates of a point in a Euclidean space of n^2 real or complex dimensions, the set of points representing the real unitary matrices (i. e. the orthogonal matrices) divides itself into two separate continua, one of which contains the rotation matrices (i. e. the orthogonal matrices of determinant $+1$) and the other the reflection matrices (i. e. the orthogonal matrices of determinant -1). The interpretation of a finite matrix as a point in a Euclidean space of a suitable number of dimensions may obviously be extended to infinite matrices by employing an abstract manifold of ∞^2 real or complex dimensions. The aim of the present paper is to obtain a division of this abstract manifold into two sub-manifolds in a manner which is a natural extension of the afore-mentioned division of the space of finite orthogonal matrices. The procedure is to first obtain a one-to-one parametric representation of the entire manifold of the infinite orthogonal matrices which divides the ortho-

gonal matrices into four distinct classes, and then to show how the matrices of these four classes may be reapportioned into two sub-manifolds of the manifold of the infinite orthogonal matrices, the two sub-manifolds coinciding, for finite orthogonal matrices, with the two separate continua noted above. Since the formula of Cayley does not yield, as we show in some detail in the next few pages, a parametric representation of the entire manifold of the orthogonal matrices, it has been found necessary to employ the spectral theory which yields a one-to-one exponential representation of the entire manifold of the orthogonal matrices. Throughout this paper we employ bold-faced capital letters to designate finite matrices, the Roman capitals being retained for infinite matrices.

We denote by

$$(1) \quad \Phi [M; x] = \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} m_{pq} x_p \bar{x}_q,$$

the *compound form* (Kopplungsform) of an arbitrary bounded† matrix $M = \| m_{pq} \|$ and shall call a real or complex vector $x = (x_1, x_2, \dots)$ a *unit vector* if

$$(2) \quad |x| = (\Phi [E; x])^{1/2} = \left(\sum_{p=1}^{+\infty} |x_p|^2 \right)^{1/2} = 1,$$

in which E is the unit matrix. The compound forms of bounded matrices possess the obvious uniqueness property in that the necessary and sufficient condition for the equality

$$\Phi [M; x] = \Phi [N; x]$$

to hold for every unit vector x is that $M = N$. Expressed otherwise, the equality

$$\Phi [M; x] = 0,$$

holds for every unit vector x then and only then if M is the zero matrix $\| 0 \|$.

We shall understand by the *domain*‡ $\Delta(M)$ of an arbitrary bounded matrix M the point-set which one constructs in the complex plane from the values, together with their limiting values, which are taken by the compound form (1) for those vectors x satisfying (2), i. e. for the totality of the unit vectors which compose the so-called complex Hilbert sphere (2). The domain $\Delta(M)$ of a bounded matrix M is then, from its definition, a bounded closed point set in the complex plane.

† D. Hilbert, "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," (1912), p. IV. This book will be referred to as "Hi".

‡ Termed by German writers "Wertevorrat."

According to a theorem of Toeplitz† there exists for every bounded matrix M at most one bounded matrix K for which the two independent conditions $KM = E$, $MK = E$ are fulfilled. If such a bounded matrix exists we designate it as the reciprocal of M and denote it by M^{-1} . We then understand by the *spectrum*‡ of a bounded matrix N the set of those points λ in the complex plane for which $(\lambda E - N)^{-1}$ does not exist and denote this point-set by $\Lambda(N)$. It is known§ that the point-set $\Lambda(N)$ thus defined for every bounded matrix N contains at least one point and is a closed subset of the domain $\Delta(N)$. If we denote by $l[M]$ the greatest lower bound of the compound form $\Phi[N; x]$ of a bounded matrix N under the condition that $|x| = 1$ and by N^* the transposed matrix of the conjugate complex elements of N (i. e. $N^* = \bar{N}'$), the number λ is then and only then, according to Toeplitz,¶ in the spectrum of N if at least one of the following two equations is fulfilled

$$(3) \quad l[(\lambda E - N)(\lambda E - N)^*] = 0, \quad l[(\lambda E - N)^*(\lambda E - N)] = 0.$$

If N is commutable with N^* , i. e. N is a normal matrix, the two conditions in (3) are identical.

The subset of the spectrum composed of those and only those points λ for which the homogeneous equations belonging to the bounded matrix $\lambda E - N$ possess a solution x for which $|x| = 1$ is called the *point spectrum* || of N . The points of the spectrum do not necessarily all belong to the point spectrum and it is possible that the point spectrum of N contains no single point. It is clear that the spectrum of $cE - N$ is obtained from the spectrum of N by a translation, determined by the real or complex number c , in the complex plane. If T is a bounded matrix for which T^{-1} exists then the bounded matrices M and TMT^{-1} possess the same spectrum and the same point spectrum.

A matrix N is said to be *Hermitian* if $N^* = N$, *unitary* if N^{-1} exists and is equal to N^* , and *skew-symmetric* if $N' = -N$. It is known that the unitary matrices form a group of bounded matrices and that no unitary

† O. Toeplitz, "Die Jacobische Transformation der quadratischen Formen von unendlichvielen Veränderlichen," *Göttingen Nachrichten* (1907), pp. 101-109. This paper will be referred to as "T".

‡ A. Wintner, "Zur Theorie der beschränkten Bilinearformen," *Mathematische Zeitschrift*, Vol. 30 (1929), p. 239. This paper will be referred to as "W".

§ See "W", pp. 241-243.

¶ See "T".

|| This definition of the point spectrum of an arbitrary bounded matrix N coincides for bounded Hermitian and unitary matrices with the usual definition for the point spectrum of matrices of these two classes.

matrix is completely continuous (vollstetig). The domain of an arbitrary bounded matrix N is a unitary invariant, i. e. if U is any unitary matrix, then $\Delta(N) = \Delta(UNU^{-1})$.

The domain $\Delta(H)$ of a bounded Hermitian matrix H is a bounded, closed interval of the real axis of the complex plane, the end-points of which always belong $\dagger$ to the spectrum $\Lambda(H)$ of H . Since the spectrum of any bounded matrix, as has been mentioned above, is a subset of the domain of the matrix, the spectrum $\Lambda(H)$ of a Hermitian matrix H lies entirely on the real axis of the complex plane. The domain of a unitary matrix U is a convex subset of the unit circle, the spectrum of U being contained on the boundary of the unit circle. The real subset of the manifold of the unitary matrices forms the manifold of the *orthogonal matrices* O so that the spectrum of O is likewise contained on the boundary of the unit circle and is in addition symmetric with respect to the axis of reals inasmuch as the spectrum of any real bounded matrix possesses this symmetry property. The domain of a bounded real skew-symmetric matrix is contained on the axis of imaginaries and, as may easily be verified directly, is symmetric with respect to the axis of reals. For our purpose it will be more instructive to derive this symmetry property of the domain of a bounded real skew-symmetric matrix as follows: The matrix iS in which S is a bounded real skew-symmetric matrix is obviously Hermitian so that its domain and, therefore, its spectrum is contained on the axis of reals. Since the spectrum of S is symmetric with respect to the axis of reals, the spectrum of iS is symmetric with respect to the axis of imaginaries and, therefore, also the domain of iS is symmetric with respect to the axis of imaginaries inasmuch as, as has been mentioned above, the end-points of the domain of a bounded Hermitian matrix H always belong to the spectrum of H . Finally, since $\Phi[S; x] = -i\Phi[iS; x]$, it follows that the domain of S is symmetric with respect to the axis of reals.

A pair of matrices (M, N) is said to be *separated* if the elements m_{pq} of M and the elements n_{pq} of N stand in the following relationship to one another:

$$(4) \quad \begin{array}{ll} \text{if } m_{pq} \neq 0 & \text{then } n_{pk} = n_{jp} = n_{jq} = n_{qk} = 0, \\ \text{if } n_{pq} \neq 0 & \text{then } m_{pk} = m_{jp} = m_{jq} = m_{qk} = 0, \end{array}$$

in which the two subscripts j, k take independently (p and q fixed) all positive integral values.

$\dagger$ F. Riesz, "Les systèmes d'équations linéaires à une infinité d'inconnues" (1913), pp. 139-140. For a direct demonstration cf. A. Wintner, "Spektraltheorie der unendlichen Matrizen" (1929), pp. 146-148. The book of Riesz will be referred to as "R" and that of Wintner as "S".

If two bounded matrices M and N form a separated pair (M, N) and if $\Phi_M \Phi_N$ are any two points whatsoever in the domains $\Delta(M)$ and $\Delta(N)$ respectively, it is clear from (4) that there exist two sequences of unit vectors so that (cf. the definition of the domain of a bounded matrix on p. 580)

$$(5) \quad \begin{aligned} \Phi_M &= \lim_{k \rightarrow +\infty} \Phi[M; y_k] \quad \text{while} \quad 0 = \Phi[N; y_k] \quad \text{for} \quad k = 1, 2, \dots, \\ \Phi_N &= \lim_{k \rightarrow +\infty} \Phi[N; z_k] \quad \text{while} \quad 0 = \Phi[M; z_k] \quad \text{for} \quad k = 1, 2, \dots, \end{aligned}$$

It is not difficult to see that if the matrix pair (M, N) is separated, the infinitely many matrix pairs

$$(6) \quad (M^p, N^q); \quad (p, q = 1, 2, \dots)$$

are also separated. A pair of bounded matrices (M, N) is said to be *perpendicular* if

$$(7) \quad MN = NM = \|0\|,$$

and it is easy to see that while a separated pair of bounded matrices is always perpendicular, it does not follow that a perpendicular pair of matrices is necessarily separated. In addition the property of a pair of bounded matrices to be perpendicular is readily seen, in contrast to the property of a matrix pair to be separated, to be invariant with respect to simultaneous matrix transformations of the two matrices composing the pair. It is trivial from (7) that for a perpendicular matrix pair (M, N) we have

$$(8) \quad (M + N)^n = M^n + N^n; \quad (n = 1, 2, \dots).$$

We employ (5), (6) and (8) in order to show that a separated pair of Hermitian matrices (M, N) possesses in the notation of p. 581 the property

$$(9) \quad \Lambda(M + N) = \Lambda(M) + \Lambda(N)$$

in which the $+$ sign denotes the usual process of the addition of two point sets, i. e. for example $\Lambda(M) + \Lambda(M) = \Lambda(M)$. In order to prove (9) let μ and ν be any two points (not necessarily different) in the respective spectra of the bounded Hermitian matrices M and N forming the separated matrix pair (M, N) . According to (3) there accordingly exist two sequences $\{y_n\}$ and $\{z_n\}$ of unit vectors for which

$$(10) \quad \begin{aligned} \lim_{n \rightarrow +\infty} \Phi[(\mu E - M)(\mu E - M)^*; y_n] \\ &= \lim_{n \rightarrow +\infty} \Phi[(\mu^2 E - 2\mu M + M^2); y_n] = 0, \\ \lim_{n \rightarrow +\infty} \Phi[(\nu E - N)(\nu E - N)^*; z_n] \\ &= \lim_{n \rightarrow +\infty} \Phi[(\nu^2 E - 2\nu N + N^2); z_n] = 0, \end{aligned}$$

and we now employ (5), (6) and (8) to show that both μ and ν lie in the spectrum of the Hermitian matrix $M + N$. In order that a point λ lies in the spectrum of $M + N$ it is, again according to (3), necessary and sufficient that there exist a sequence of unit vectors $\{x_n\}$ for which

$$(11) \quad \lim_{n \rightarrow +\infty} \Phi [(\lambda E - M - N)(\lambda E - M - N)^*; x_n] \\ = \lim_{n \rightarrow +\infty} \Phi [(\lambda^2 E - 2\lambda M - 2\lambda N + M^2 + N^2); x_n] = 0,$$

inasmuch as from (8) we have $(M + N)^2 = M^2 + N^2$. Since, according to (6), the matrix pairs (M, N) , (M, N^2) , (M^2, N) , (M^2, N^2) are all separated, the condition (11) may obviously, from (10), be satisfied by putting either

$$\lambda = \mu \text{ and } \{x_n\} = \{y_n\} \quad \text{or} \quad \lambda = \nu \text{ and } \{x_n\} = \{z_n\},$$

inasmuch as, as follows from (5) and (6), the sequences $\{y_n\}$ and $\{z_n\}$ in (10) may be chosen to have

$$\Phi [M; z_n] = 0, \Phi [M^2; z_n] = 0; \quad \Phi [N; y_n] = 0, \Phi [N^2; y_n] = 0,$$

for $n = 1, 2, \dots$, i. e. the points μ and ν both belong to the spectrum of $M + N$.

An important application for p. 607 of the text concerns itself with the special case in which M is a diagonal matrix, the elements of which are either 0 or $+1$. The complementary matrix $E - M$ is accordingly of the same type as M and forms with M the separated pair $(E - M, M)$. We then construct the separated pair $(c[E - M] + N, cM)$ where c is any number, real or complex, and apply the result (9) of the preceding paragraph to this matrix pair, thereby obtaining

$$\Lambda(cE + N) = \Lambda(c[E - M] + N + cM) = \Lambda(c[E - M] + N) + \Lambda(cM),$$

or, in particular for $c = \pi$ and $N = iS$

$$(12) \quad \Lambda(\pi M) + \Lambda(\pi[E - M] + iS) = \Lambda(\pi E + iS).$$

Since the spectrum $\Lambda(\pi M)$ of πM only contains the two points 0 and π , it is clear from (12) that $\lambda > \pi$ is then and only then a point in the spectrum of $\pi[E - M] + iS$ if it is a point in the spectrum of $\pi E + iS$. It is furthermore trivial that 2π is then and only then in the point spectrum of $\pi E + iS$ if $+\pi$ (and if S is real and skew-symmetric then also $-\pi$) is a point in the point spectrum of iS so that we are led to the following result employed on p. 610 of the text: If S is a real skew-symmetric matrix whose spectrum is contained in the closed interval $[-i\pi, +i\pi]$ and if M is a diagonal matrix, the elements of which are either 0 or $+1$, forming with S the

separated matrix pair (M, S) , then the point 2π lies in the point spectrum of the Hermitian matrix $\pi(E - M) + iS$ if, and only if, if the end-points $-\pi$ and $+\pi$ of the interval $[-\pi, +\pi]$ belong to the point spectrum of iS .

A matrix I is said to be an *idempotent* matrix if $I^2 = I$ (inasmuch as then $I^n = I$ for $n = 1, 2, \dots$). The Hermitian idempotent matrices are designated by Hilbert as "Einzelmatrizen." The bounded idempotent matrices I possess the property that the continuously many matrices $2\pi inI$ ($n = 0, \pm 1, \dots$) are all "logarithms" of E , i. e. $\exp(2\pi inI) = E$ [Cf. (80)]. Since the matrix

$$[(\lambda - 1)E + I]/[\lambda(\lambda - 1)]$$

represents, as may easily be verified, for $\lambda(\lambda - 1) \neq 0$ the reciprocal matrix of $(\lambda E - I)$ for every bounded idempotent matrix I , the spectrum of I contains at most the two points $\lambda = 0$ and $\lambda = +1$. For real symmetric (and therefore bounded) idempotent matrices there exists, according to Hilbert,[†] an orthogonal matrix O so that $OIO^{-1} = \|\kappa_{pq}\|$ where $\kappa_{pq} = 0$ for $p \neq q$ and $\kappa_{pp} = 0$ or $\kappa_{pp} = +1$ (Cf. the matrices M and $E - M$ above). We denote by α' the number of times 0 occurs in the sequence $\{\kappa_{pp}\}$ and by β' the number of times $+1$ occurs in this sequence so that $\alpha' \geq 0$, $\beta' \geq 0$ and $\alpha' + \beta' = +\infty$. The numbers α' and β' are uniquely determined by the matrix I , i. e. they are independent of the special choice of the matrix O employed in transforming I into the diagonal form $\|\kappa_{pq}\|$. It will be necessary to distinguish the following four classes of idempotent matrices:

- | | | | | |
|------|----------------------|---------------------|-----|-----------------------|
| I: | $\alpha' = +\infty,$ | $\beta' = 2m$ | for | $0 \leq m < +\infty,$ |
| II: | $\alpha' = +\infty,$ | $\beta' = 2m + 1$ | for | $0 \leq m < +\infty,$ |
| III: | $\alpha' = m,$ | $\beta' = +\infty$ | for | $0 \leq m < +\infty,$ |
| IV: | $\alpha' = +\infty,$ | $\beta' = +\infty.$ | | |

We call a matrix Q a *square root* of E if $Q^2 = E$. The bounded square roots of E may be characterized by the matrix equation $Q^{-1} = Q$ and possess the property that the continuously many matrices $2\pi inQ$ ($n = 0, \pm 1, \pm 2, \dots$) are all "logarithms" of E (Cf. above). The general real symmetric square root of E depends (as does the general real symmetric idempotent matrix) upon infinitely many arbitrary constants. Since $(\lambda E + Q)(\lambda^2 - 1)^{-1}$ represents, as may easily be verified, for $\lambda^2 - 1 \neq 0$ the reciprocal matrix of $\lambda E - Q$ for every bounded square root Q of E , the spectrum of Q contains at most the two points $\lambda = +1$ and $\lambda = -1$. It therefore follows from

[†] See "Hi", p. 145.

the theorem of Hilbert † on bounded real symmetric matrices without continuous spectra that there exists for every real symmetric (and therefore bounded) square root of E an orthogonal matrix O so that $OQO^{-1} = \|\epsilon_{pq}\|$ where $\epsilon_{pq} = 0$ for $p \neq q$ and $\epsilon_{pp} = +1$ or $\epsilon_{pp} = -1$. We denote by α the number of times $+1$ occurs in the sequence $\{\epsilon_{pp}\}$ and by β the number of times -1 occurs in this sequence so that $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta = +\infty$. The numbers α and β are uniquely determined by the matrix Q , i. e. they are independent of the special choice of the matrix O employed in transforming Q into the diagonal form $\|\epsilon_{pq}\|$. It will be necessary to distinguish the following four classes of matrices which are square roots of E :

$$(13) \quad \begin{array}{lll} \text{I:} & \alpha = +\infty, & \beta = 2m \quad \text{for } 0 \leq m < +\infty, \\ \text{II:} & \alpha = +\infty, & \beta = 2m + 1 \quad \text{for } 0 \leq m < +\infty, \\ \text{III:} & \alpha = m, & \beta = +\infty \quad \text{for } 0 \leq m < +\infty, \\ \text{IV:} & \alpha = +\infty, & \beta = +\infty. \end{array}$$

For the sake of brevity we shall denote an orthogonal matrix not containing -1 in its spectrum as a *Cayley matrix*. That not every orthogonal matrix is a Cayley matrix is trivial,‡ even for finite matrices; for example, no single reflection matrix (i. e. an orthogonal matrix of determinant -1) is a Cayley matrix. The justification for the name of this sub-manifold of the manifold of the orthogonal matrices is contained in the statement that the sub-manifold of the Cayley matrices may be put in a one-to-one correspondence with the manifold of the real bounded skew-symmetric matrices by means of the well known formula of Cayley,

$$(14) \quad O = (E + S)(E - S)^{-1} = (E - S)^{-1}(E + S).$$

† See "Hi", p. 145.

‡ Cf. M. Born and P. Jordan, "Elementare Quantenmechanik" (1930), p. 37, in which they point out that the representation

$$U = (E + iH)(E - iH)^{-1} = (E - iH)^{-1}(E + iH),$$

in which H is a bounded Hermitian matrix, is restricted to those unitary matrices U not containing -1 in their spectrum. The matrix U in this representation is then and only then a real matrix, i. e. an orthogonal matrix if $H = iS$, where S is a bounded real skew-symmetric matrix, the representation becoming, in this case, the well known representation of Cayley. For finite orthogonal matrices it is not true, as is sometimes stated, [cf. for instance H. W. Turnbull, "The Theory of Determinants, Matrices and Invariants" (1928), pp. 155-157] that every orthogonal matrix of determinant $+1$ permits the Cayley representation (cf. in particular p. 588). On the other hand, those orthogonal matrices of determinant $+1$, not permitting the Cayley representation, may be regarded as limiting cases of orthogonal matrices permitting the Cayley representation. In this connection cf. L. E. Dickson, "Modern Algebraic Theories" (1926), p. 102.

In order to demonstrate this let S in (14) be a bounded real skew-symmetric matrix so that $(E + S)^{-1}$ and $(E - S)^{-1}$ exist. The matrix O in (14) is then obviously real and is orthogonal for

$$OO' = O'O = (E + S)(E - S)^{-1}(E - S)(E + S)^{-1} = E.$$

Furthermore it is a Cayley matrix for, from (14),

$$\begin{aligned} O + E &= (E + S)(E - S)^{-1} + E = (E + S)(E - S)^{-1} \\ &+ (E - S)(E - S)^{-1} = (E + S + E - S)(E - S)^{-1} = 2(E - S)^{-1} \end{aligned}$$

so that $(O + E)^{-1}$ exists, i. e. -1 is not a point in the spectrum of O . For the converse proposition, i. e. to any given Cayley matrix O there always exists a real bounded skew-symmetric matrix S for which (14) is valid, we consider the matrix

$$(15) \quad S = (O - E)(O + E)^{-1},$$

which may be constructed for every Cayley matrix O . We first observe that because of

$$(O + E)(O - E) = (O - E)(O + E),$$

i. e.

$$(O - E)(O + E)^{-1} = (O + E)^{-1}(O - E),$$

we also have

$$(16) \quad S = (O + E)^{-1}(O - E).$$

Now the matrix S in (15), (16) is certainly real and bounded. In addition it is skew-symmetric, for

$$\begin{aligned} S' &= (O' + E)^{-1}(O' - E) = (O' + E)^{-1}O'O(O' - E) \\ &= [O(O' + E)]^{-1}(E - O) = (O + E)^{-1}(E - O) = -S, \end{aligned}$$

and may be employed as S in the representation (14) of O since $(E + S)(E - S)^{-1}$

$$\begin{aligned} &= [E + (O - E)(O + E)^{-1}](O + E)(O + E)^{-1}[E - (O - E)(O + E)^{-1}]^{-1} \\ &= (O + E + O - E)(O + E - O + E)^{-1} = O. \end{aligned}$$

That this correspondence set up between the manifold of the Cayley matrices and the bounded real skew-symmetric matrices is one to one is trivial. One may see, even for finite matrices, that the manifold of the Cayley matrices is only a very restricted subset of the manifold of all orthogonal matrices. For finite matrices the manifold of the Cayley matrices is only a sub-manifold of the manifold of the rotation matrices, not even every rotation matrix being a Cayley matrix. As an example of a matrix which is a rotation matrix and

not a Cayley matrix one has the matrix $-\mathbf{E}$ which for even n is a rotation matrix, but is for no single value of n a Cayley matrix. This fact is emphasized very forcibly by actually attempting the representation (14) for $\mathbf{O} = -\mathbf{E}$, thereby obtaining

$$-\mathbf{E} = (\mathbf{E} + \mathbf{S})(\mathbf{E} - \mathbf{S})^{-1},$$

and consequently

$$-\mathbf{E} + \mathbf{S} = \mathbf{E} + \mathbf{S} \quad \text{so that} \quad -\mathbf{E} = \mathbf{E},$$

an obvious contradiction. We are accordingly convinced that, although the Cayley representation (14) possesses no convergence difficulties, it is impossible to obtain, by its use, a parametric representation of the entire manifold of orthogonal matrices. In order to obtain a parametric representation of the entire manifold of the orthogonal matrices we employ the spectral theory which will yield a canonical exponential representation.

A Hermitian matrix is said to be *reduced* if its spectrum lies in the closed interval $[0, 2\pi]$ and if its point spectrum does not contain the point 2π . According to Wintner[†] a one to one correspondence is set up between the manifold of the unitary matrices and the manifold of the reduced Hermitian matrices by means of the exponential representation

$$(17) \quad \mathbf{U} = e^{i\mathbf{H}},$$

in which $\mathbf{U}$ is a unitary matrix and $\mathbf{H}$ a reduced Hermitian matrix. The definition of the matrix $e^{\mathbf{M}}$ is of course

$$e^{\mathbf{M}} = \sum_{\nu=0}^{\infty} (1/\nu!) \mathbf{M}^{\nu},$$

which represents for every bounded matrix $\mathbf{M}$ a bounded matrix. Consequently if $(\mathbf{M}, \mathbf{N})$ is a commutable pair of bounded matrices, we have

$$(18) \quad e^{\mathbf{M}+\mathbf{N}} = e^{\mathbf{M}} \cdot e^{\mathbf{N}} = e^{\mathbf{N}} \cdot e^{\mathbf{M}},$$

and if $(\mathbf{M}, \mathbf{N})$ is not only a commutable pair of matrices but also perpendicular [Cf. (7)]

$$(19) \quad e^{\mathbf{M}+\mathbf{N}} = e^{\mathbf{M}} + e^{\mathbf{N}} - \mathbf{E}.$$

For our purposes it is necessary to place the manifold of the bounded Hermitian matrices in a one-to-one correspondence with the combined manifolds

[†] See "W", pp. 268-277. For further literature cf. H. Weyl, *The Theory of Groups and Quantum Mechanics* (1931), p. 399.

of the bounded real symmetric and real skew-symmetric matrices by means of the linear representation

$$(20) \quad H = \pi B + iS, \quad B = \bar{B} = B', \quad S = \bar{S} = -S'.$$

In Theorem I we show that from a consideration of a trigonometric momentum problem with real "momenta" the entire manifold of the orthogonal matrices may, by means of the exponential representation (17), (20), i. e.

$$(21) \quad O = e^{iH} = e^{i[\pi B + iS]},$$

be put in a one to one correspondence $\dagger$ with that sub-manifold of the reduced Hermitian matrices for which in (20)

$$(22) \quad B^2 = B, \quad BS = SB,$$

i. e. B is a real symmetric idempotent (and therefore bounded) matrix and S a bounded real skew-symmetric matrix, commutable with B , and such that $\pi B + iS$ is a reduced Hermitian matrix.

On p. 606 we introduce the term *reduced skew-symmetric matrix* for those skew-symmetric matrices S , the entire spectrum of which is contained in the closed interval $[-i\pi, +i\pi]$ of the imaginary axis, and whose point spectrum does not contain the end-points $-i\pi$ and $+i\pi$. We then show in Lemma I that the manifold of the reduced Hermitian matrices (20), (22) may be put in a one-to-one correspondence with the manifold of the perpendicular matrix pairs $(E - B, S)$ in which S is a reduced skew-symmetric matrix. As a consequence of Lemma I we then obtain Theorem II which states that the entire manifold of the orthogonal matrices may, by means of the exponential representation (21), be put in a one-to-one correspondence with the manifold of the matrix pairs (B, S) where

$$(23) \quad B = \bar{B} = B', \quad B^2 = B; \quad S = \bar{S} = -S'; \\ (E - B)S = S(E - B) = \|0\|, \quad S \text{ is reduced.}$$

We then show in Theorem III that the one-to-one representation (21), (22) is in reality none other than the one-to-one representation

$$(24) \quad O = 2(E - B) - e^{-S}$$

in which the matrix pair (B, S) is identical with the matrix pair (B, S) in (21), (22).

In Lemma II on p. 615, we show that the manifold of the matrices Q which are square roots of E may be placed in a one-to-one correspondence

$\dagger$ See "W", p. 277.

with the manifold of the idempotent matrices I by means of the linear representation

$$(25) \quad Q = E - 2I.$$

An application of this lemma to Theorem I then yields Theorem IV on p. 615, in which the entire manifold of the orthogonal matrices is shown to be in a one-to-one correspondence, by means of the representation

$$(26) \quad O = Q \cdot e^{-S} = e^{-S} \cdot Q$$

with the manifold of the matrix pairs (Q, S) in which Q is a real symmetric square root of E (and therefore itself an orthogonal matrix) and S a bounded real skew-symmetric matrix, commutable with Q , and such that the Hermitian matrix $(\pi/2)(E - Q) + iS$ is reduced.

Since Theorem IV assigns to every orthogonal matrix O a uniquely determined real symmetric square root Q of E we may interpret the classification (13) of the real symmetric square roots of E as a classification of the orthogonal matrices into the four classes of p. 617. As immediate examples of the four types of orthogonal matrices we have the matrices Q of (124) themselves.

For normal forms, under orthogonal matrix transformations of the matrices of each of the four classes we obtain, as consequences of Theorems I and IV and Lemma I, the canonical forms given in (133) in which the matrices $Q^{(i)}$ and $S^{(i)}$ ($i = 1, 2, 3, 4$) are defined in (124) and (125) respectively. The matrices O_{2m} and O_{2m+1} occurring in I and II of (133) are the canonical finite rotation matrices as defined in (131). From these canonical forms I and II of (133) it is clear at once that the orthogonal matrices of Classes I and II can be interpreted, with respect to a suitably orientated coördinate system, as rotations and reflections (in the geometrical sense of the word) in some subspace of the real Hilbert space, possessing a sufficiently high but finite number of dimensions, in which a subspace of infinitely many dimensions is not "affected" † by the transformation. The orthogonal matrices of Class III may accordingly be interpreted as representing those orthogonal transformations of the real Hilbert space which leave a subspace of at most a finite number of dimensions "unaffected" and the orthogonal matrices of Class IV as the orthogonal transformations of the real Hilbert space which

† For example we say that the transformation in the ordinary Euclidean 3-space possessing the matrix

$$\begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

leaves a space of one dimension not "affected" by the transformation.

leave "unaffected" a subspace of infinitely many dimensions while simultaneously "affecting" a complement subspace of infinitely many dimensions. It may not be superfluous to remark that according to this classification the unit matrix E is an orthogonal matrix of Class I and the orthogonal matrix $-E$ of Class III. It is still an open question as to whether the orthogonal matrices of Classes III and IV may be divided into "rotations" and "reflections." In this connection cf. p. 592, where we are able to give a precise formulation of this unsolved problem.

During the last few years several authors have investigated rotations of the functional space, or what is the same, rotations in the Hilbert space. These papers, however, deal only with very special orthogonal matrices, supposing, for instance, that the determinant of the matrix is convergent and that the "deviation" of the orthogonal matrix from the unit matrix is a completely continuous (vollstetig in the sense of Hilbert) matrix. Even these very restrictive conditions are not sufficient for the validity of the determinant apparatus as employed by them. In this paper we deal with the totality of the orthogonal transformations of the Hilbert space and are, therefore, denied any recourse to the apparatus of determinants. For example the inaptness of the determinant apparatus is clearly revealed by attempting to differentiate, by its use, between the various classes (133) of the orthogonal matrices $Q^{(1)}$ of (124).

The Classes I and II of the orthogonal matrices O are then shown to be the natural extensions of the finite rotation and reflection matrices as defined on p. 579. The matrices of Classes III and IV obviously have no analogues for finite matrices. At this point there is introduced a definition which divides† the entire manifold of the infinite orthogonal matrices into two sub-manifolds in such a manner that the definition when applied to the manifold of the finite orthogonal matrices divides it into two sub-manifolds which are identical with the sub-manifolds of the finite rotation and reflection matrices as defined on p. 579. We define on p. 620 that sub-manifold of the orthogonal matrices, the elements O of which may be represented in the form e^G where G is a real bounded skew-symmetric matrix [not necessarily uniquely determined by O (Cf. p. 594)], i. e.

$$(27) \quad O = e^G, \quad G = \bar{G} = -G',$$

as the manifold of the *rotation matrices* and the complement of this manifold with respect to the entire manifold of the orthogonal matrices [i. e. the manifold of those matrices not permitting the exponential representation (27)],

† Cf. p. 579.

as the manifold of the *reflection matrices*. This definition of the manifolds of the rotation and reflection matrices will be referred to as the exponential definition in order to distinguish it from the determinant definition on p. 579 which is utterly valueless for infinite orthogonal matrices since the determinant of an orthogonal matrix is convergent only for very special orthogonal matrices. In order to justify this definition we show on p. 620 in Lemma III that the exponential definition of the rotation and reflection matrices is, for finite matrices, identical with the determinant definition. In Theorem V on p. 621, we show that the matrices of Classes I and II are, in the sense of the exponential definition, respectively rotation and reflection matrices. The first part of Theorem V follows directly from the canonical form for the matrices of Class I. In order to prove the second part of Theorem V we show in Lemma IV on p. 622 that if θ (θ real) is a point in the spectrum of an arbitrary (not necessarily reduced) Hermitian matrix H then $e^{i\theta}$ is a point in the spectrum of e^{iH} . As a corollary to Lemma IV we have on p. 624 that if $i\theta$ is a point in the spectrum of an arbitrary real bounded skew-symmetric matrix S then $e^{i\theta}$ is a point in the spectrum of e^S from which, and the canonical form II of (133), the second part of Theorem V is shown to follow.

The unsolved problem, mentioned on p. 591, as to whether the orthogonal matrices of Classes III and IV permit a resolution into "*rotations*" and "*reflections*" may now be put in a precise form as follows: Do the Classes III and IV contain both rotation and reflection matrices as defined in (27)? That there exist rotation matrices in the sense of the exponential definition (27) is trivial since the orthogonal matrices $Q^{(3)}$ and $Q^{(4)}$ of (124) are of Classes III and IV respectively and obviously permit the exponential representation (27). The unsolved part of the problem then requires the demonstration of the existence or the non-existence of reflection matrices, as defined in (27), for each of the Classes III and IV. Put more directly, do the matrices of the Classes III and IV all permit the exponential representation (27)? Since the property of a matrix to be a rotation matrix, in the sense of the exponential definition (27), is obviously invariant[†] with respect to orthogonal transformations, it is sufficient to treat this question for the canonical forms III and IV of (133) for the matrices of Classes III and IV. From the fact that the orthogonal matrices $Q^{(3)}$ and $Q^{(4)}$ in the canonical forms (133) are both rotation matrices, it is trivial that if the

[†] This invariance property is trivial inasmuch as follows from the definition of eG on p. 13 that if O is an arbitrary orthogonal matrix then $OeGO^{-1} = eOGO^{-1}$ in which OGO^{-1} is again a bounded real skew-symmetric matrix.

rotation matrices [i. e. those orthogonal matrices permitting the representation (27)] form a group, the canonical matrices III and IV of (133) are necessarily rotation matrices [i. e. also permit the representation (27)]. It may not be superfluous to point out that, because of the special character of the matrices occurring in the canonical matrices III and IV of (133), the condition that the rotation matrices (27) form a group is not a necessary condition for the matrices of Classes III and IV to be rotation matrices (27). A treatment of the above unsolved problems seem to require a detailed investigation of orthogonal and real skew-symmetric matrices possessing continuous spectra. Connected with this problem is the question † as to whether the manifold of all orthogonal matrices forms a connected manifold or not (with respect to some natural metric in the manifold of all matrices). For the orthogonal transformations of a space with a finite number of dimensions the manifold of all orthogonal matrices is not connected (with respect to every natural metric of the manifold) but is divided into two separate manifolds (Cf. p. 579). In other words the Euclidean space of a finite number of dimensions may be said to be an "orientable" space. It has not been possible as yet to answer the question, on the basis of the parametrical representation of orthogonal matrices given in the paper, as to whether the real Hilbert space forms, in this sense, an orientable space or not. In order that the manifold of all orthogonal matrices be connected it is necessary that each of the sub-manifolds in (133) be connected. It seems likely that for the answer to the above question the extensive use of the theory of the spectral matrix of the orthogonal matrices is unavoidable, the problems treated in the present paper depending upon the theory of continuous spectra of unitary matrices (cf. "W", pp. 257-258).

Theorems I, II, III and IV all give a complete parametric representation of the manifold of the orthogonal matrices and we saw that (27) where G is an arbitrary real bounded skew-symmetric matrix, may be defined as the parametric representation of the manifold of the rotation matrices of the Hilbert space. This parametric representation of the rotation matrices is the substitute † in Hilbert space for the Euler-Hurwitz factorization ‡ into products of "two-block" rotation matrices which breaks down for infinite

† See "W", p. 277.

‡ L. Euler, *Commentationes arithmeticae collectae*, Vol. 1, p. 427 and A. Hurwitz, "Ueber die Erzeugung der Invarianten durch Integration," *Göttingen Nachrichten* (1897), pp. 76-77. In connection with the work of Hurwitz, cf. I. Schur, "Neue Anwendungen der Integralrechnung auf Probleme der Invariantentheorie," *Sitzungsberichte der Preussischen Akademie der Wissenschaften* (1924), p. 196, where a statement of Hurwitz is corrected.

matrices since there exists no last dimension. There are even other obstacles to a generalization of the Euler-Hurwitz representation inasmuch as the following example shows that a matrix which is (even in the sense of a "weak" convergence) the limit of a sequence of orthogonal "two block" matrices is not necessarily itself an orthogonal matrix. Denote by E_{pq} the "two-block" orthogonal matrix obtained from the infinite unit matrix E by interchanging the p -th and q -th columns (or what amounts to the same thing, by interchanging the p -th and q -th rows). Consider the sequence of orthogonal matrices $\{O_i\}$ defined by $O_1 = E$, $O_i = O_{i-1}E_{i-1i}$ ($i = 2, 3, \dots$), i. e. the sequence

$$O_1 = E, \quad O_2 = E_{12}, \quad O_3 = E_{12}E_{23}, \dots,$$

for which

$$\lim_{n \rightarrow +\infty} O_n = \left\| \begin{array}{ccccccc} 0 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{array} \right\|,$$

and which is certainly not an orthogonal matrix.

It may be mentioned that to a given rotation matrix O there may exist infinitely many real bounded skew-symmetric matrices which may be employed as the matrix G in (27). This is illustrated, even for finite matrices, by the example

$$\left\| \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right\| = \exp \left\| \begin{array}{cc} 0 & (\theta + 2k\pi) \\ -(\theta + 2k\pi) & 0 \end{array} \right\|; \quad (k = 0, \pm 1, \dots).$$

It would be desirable to obtain a simple sub-manifold of the matrices G in (27) so that the matrices of this sub-manifold are in one-to-one correspondence with the rotation matrices. Certain considerations for finite matrices seem to indicate that this may be achieved for finite matrices by restricting the representation (27) to a sub-manifold analogous to the manifold of the reduced skew-symmetric matrices, and it is hoped to treat this question among others in a later appearing paper on the finite matrices.

For finite matrices it is possible † to obtain, for $n = 2$ and $n = 3$, the formulas of Rodrigues from the exponential representation (27).

† Cf. G. Galanti, "Algoritmi di calcolo motoriale," *Atti Accademia Nazionale dei Lincei*, Vol. 6, series 13 (1931), pp. 861-866.

Finally, it may be mentioned that the methods and results of the present paper hold, precisely as the spectral theory of unitary transformations or the method of trigonometric momentum problem used in this paper, not only for matrices but also for operators in the Hilbert space. However, since the possibility of such generalizations is known to be trivial we prefer, for clearness, to deal with matrices only.

I. REALITY DISCUSSION OF THE TRIGONOMETRICAL MOMENT PROBLEM OF HERGLOTZ.

We consider the trigonometrical moment problem †

$$(28) \quad 2\pi c_n = \int_0^{2\pi} e^{in\phi} d\tau(\phi); \quad (n = 0, \pm 1, \dots),$$

together with its homogeneous special case

$$(28') \quad 0 = \int_0^{2\pi} e^{in\phi} d\tau(\phi); \quad (n = 0, \pm 1, \dots),$$

in which the left-hand members are preassigned real or complex numbers and $\tau(\phi)$ an unknown real or complex function of the real argument ϕ whose definition is desired in the closed interval $0 \leq \phi \leq 2\pi$. In the following we shall consider only real solutions $\tau(\phi)$ and therefore suppose that

$$(29) \quad c_n = \bar{c}_{-n}; \quad (n = 0, 1, \dots).$$

It will be convenient to employ the notations

$$(30) \quad \begin{aligned} \Delta_{\beta-0} f(\phi) &= f(\beta) - f(\beta-0), \\ \Delta_{\beta} f(\phi) &= f(\beta+0) - f(\beta-0), \\ \Delta_{\beta+0} f(\phi) &= f(\beta+0) - f(\beta), \end{aligned}$$

(provided these limits exist) for the various "springs" of an arbitrary real or complex function $f = f(\phi)$ at the point $\phi = \beta$ and the customary notations $[a, b]$ and (a, b) for the closed and open intervals $a \leq \phi \leq b$ and $a < \phi < b$ respectively.

We suppose that the moment problem (28) permits a solution $\tau(\phi)$ of bounded variation in the interval of its definition $[0, 2\pi]$ so that the limits

† G. Herglotz, "Über Potenzreihen mit positiven, reellen Teil im Einheitskreis," *Königlich Sächsische Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physische Klasse* (1911), p. 501-511. This paper will be referred to as "He".

$$(31) \quad \tau(\phi - 0) \quad \text{for } 0 < \phi \leq 2\pi; \quad \tau(\phi + 0) \quad \text{for } 0 \leq \phi < 2\pi,$$

exist. It is known[†] from the general theory of Stieltjes integrals that if (28) possesses a solution $\tau(\phi)$ which is of bounded variation in the interval of its definition $[0, 2\pi]$ it is uniquely determined, up to an additive constant, on the set of its continuity points contained in the interval $(0, 2\pi)$, its values at its inner discontinuity points (i. e. for $0 < \phi < 2\pi$) being assignable at will. Since the set of continuity points of $\tau(\phi)$ in the interval $(0, 2\pi)$ is necessarily everywhere dense in this interval the values of $\tau(0 + 0)$ and $\tau(2\pi - 0)$ may then be calculated [cf. (31)] from the values of $\tau(\phi)$ at its continuity points and the values of $\tau(0)$ and $\tau(2\pi)$ chosen at will, provided that the conditions

$$(32) \quad \Delta_{0+0} \tau(\phi) + \Delta_{2\pi-0} \tau(\phi) = 2\pi c_0 - \int_{0+0}^{2\pi-0} d\tau(\phi) = 2\pi c_n - \int_{0+0}^{2\pi-0} e^{in\phi} d\tau(\phi);$$

$$(n = 0, \pm 1, \dots),$$

which arise from (28), are satisfied.[‡] The only solutions of bounded variation of the homogeneous moment problem (28') may in turn be completely characterized by the properties §

$$(33) \quad \tau(\phi) = \alpha = \bar{\alpha} = \text{const. at the continuity points of } \tau(\phi) \text{ in } (0, 2\pi),$$

$$(34) \quad \Delta_{0+0} \tau(\phi) + \Delta_{2\pi-0} \tau(\phi) = 0,$$

inasmuch as, in (32), we have $c_0 = 0$ and $\int_{0+0}^{2\pi-0} d\tau(\phi) = 0$.

A further possible specialization of the character of the solution $\tau(\phi)$ of (28) is that it be monotone non-decreasing. The necessary and sufficient conditions on the c_n in order that the moment problem (28) possess a monotone non-decreasing solution $\tau(\phi)$ are given by Herglotz ¶ and state that the truncated matrices (Abschnittsmatrizen) of the infinite Hermitian matrix $\|c_{\mu-\nu}\|$, introduced by Toeplitz, be not negative definite, i. e.

$$(35) \quad \sum_{\mu=1}^n \sum_{\nu=1}^n c_{\mu-\nu} \xi_{\mu} \bar{\xi}_{\nu} \geq 0 \text{ for arbitrary } \xi_1, \xi_2, \dots; \quad (n = 1, 2, \dots),$$

where $c_{\mu-\nu} = c_k = \bar{c}_{-k}$ for $k = \mu - \nu$.

[†] Cf. F. Hausdorff, "über das Momentenproblem für ein endliches Intervall," *Mathematische Zeitschrift*, Vol. 16 (1923), pp. 220-248. This paper will be referred to as "Ha". Cf. also O. Perron, "Die Lehre von den Kettenbrüchen" (1913), pp. 362-374. This book will be referred to as "P".

[‡] See "W", p. 268.

§ See "Ha", p. 241.

¶ See "He".

We denote by $E = \|\delta_{pq}\|$ the unit matrix and by $\Phi[A; x]$ the compound form (Kopplungsform)

$$(36) \quad \Phi[A; x] = \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} a_{pq} x_p \bar{x}_q,$$

of an arbitrary bounded matrix $A = \|a_{pq}\|$. With these notations we consider the trigonometric moment problem $\dagger$ arising from (28) if one places

$$(37) \quad c_n = \Phi[U^n; x]; \quad (n = 0, \pm 1, \dots),$$

i. e., the moment problem

$$(38) \quad 2\pi\Phi[U^n; x] = \int_0^{2\pi} e^{in\phi} d\tau(\phi; x), \quad |x| = 1; \quad (n = 0, \pm 1, \dots),$$

where U is a given unitary matrix so that for every unit vector x [i. e., $|x| = 1$] there belongs the sequence of momenta c_k defined in (37) satisfying the conditions (29). It is known that this moment problem always possesses a real solution $\tau(\phi; x)$ which depends upon the vector x and is, as a matter of fact, a bounded Hermitian form

$$(39) \quad \tau(\phi; x) \equiv \Phi[\|\tau_{pq}(\phi)\|; x],$$

where

$$\tau_{pq}(\phi) = \bar{\tau}_{qp}(\phi), \quad 0 \leq \phi \leq 2\pi,$$

and the indices p, q take, as will always be understood in the remainder of the paper, unless explicitly stated to the contrary, independently the values $1, 2, \dots$. The Hermitian form (39) is then, for an arbitrary given unit vector x , uniquely determined up to an additive constant at all of its continuity points in the interval $(0, 2\pi)$. Moreover, since the conditions (35) are known to be fulfilled by the momenta (37), for every unit vector x , the Hermitian form (39) may be chosen to be, for an arbitrary given unit vector x , a monotone non-decreasing function of ϕ in the interval $(0, 2\pi)$. The values $\tau(0, x)$ and $\tau(2\pi, x)$ of the Hermitian form at the end points may then, for an arbitrary given unit vector x , be assigned at will subject to the condition that $\tau(\phi; x)$ be monotone in the closed interval $[0, 2\pi]$ and that

$$(40) \quad \Delta_{0+0} \tau(\phi; x) + \Delta_{2\pi-0} \tau(\phi; x) = 2\pi - \int_{0+0}^{2\pi-0} d\tau(\phi; x).$$

We now remove the arbitrary character of the solution $\tau(\phi; x)$ by prescribing, for an arbitrary given unit vector x , the following normalizing conditions:

$\dagger$ See "W", p. 269.

$$(41) \quad \tau(0; x) \equiv 0,$$

$$(42) \quad \tau(\phi - 0; x) \equiv \tau(\phi; x) \quad \text{for } 0 < \phi < 2\pi,$$

$$(43) \quad \Delta_{2\pi-0} \tau(\phi; x) \equiv 0.$$

The normalizing condition (42) serves to fix the value of the solution $\tau(\phi; x)$ at its discontinuity points in the interval $(0, 2\pi)$. The prescribing of the normalizing condition (43) then from (40) serves to determine uniquely the "spring" $\Delta_{0+0} \tau(\phi; x)$. Finally the normalizing condition (41) fixes the value of the additive constant so that the solution $\tau(\phi; x)$ is uniquely determined at all points of the closed interval $[0, 2\pi]$ since, for $\phi = 2\pi$, we have from (41) and (38), for $n = 0$,

$$(44) \quad \tau(2\pi; x) \equiv 2\pi, \quad |x| = 1.$$

The matrix of the Hermitian form under the normalizing conditions (41), (42) and (43) has been introduced by Wintner † and designated by him as the spectral matrix of U . Since (38), (41), (42), (43) and (44) hold for an arbitrary unit vector x the elements of the spectral matrix $\|\tau_{pq}(\phi)\|$ obviously possess the following properties ‡:

$$(38') \quad 2\pi U^n = \left\| \int_0^{2\pi} e^{in\phi} d\tau_{pq}(\phi) \right\|; \quad (n = 0, \pm 1, \dots),$$

$$(41') \quad \tau_{pq}(0) = 0,$$

$$(42') \quad \tau_{pq}(\phi - 0) = \tau_{pq}(\phi) \quad \text{for } 0 < \phi < 2\pi,$$

$$(43') \quad \Delta_{2\pi-0} \tau_{pq}(\phi) = 0 \quad \text{i. e.} \quad \tau_{pq}(2\pi - 0) = \tau_{pq}(2\pi),$$

$$(44') \quad \tau_{pq}(2\pi) = 2\pi\delta_{pq}.$$

If now the unitary matrix is assumed to be real, i. e. U is an orthogonal matrix $U = \bar{U} = O$, the elements of the spectral matrix possess in addition to the above mentioned properties the following two properties:

I. In order that the elements $\tau_{pq}(\phi)$ of the spectral matrix of the unitary matrix U be the elements of the spectral matrix of a real unitary matrix $U = \bar{U}$, i. e. an orthogonal matrix O , it is necessary and sufficient that

$$(45) \quad \omega_{pq}(\phi) = b\delta_{pq}$$

at the continuity points of $\omega_{pq}(\phi)$ in $(0, 2\pi)$, where

† See "W", p. 269.

‡ Cf. the uniqueness property of compound forms mentioned on p. 580.

$$(46) \quad \omega_{pq}(\phi) = \bar{\tau}_{pq}(\phi) + \tau_{pq}(2\pi - \phi) \quad \text{and} \quad b = \bar{b} = \text{const.}$$

II. The “springs” $\Delta_{0+0} \tau_{pq}(\phi)$ of the elements of the spectral matrix of an orthogonal matrix O are real.

The demonstration of the two statements I and II proceeds simultaneously and we begin by showing the necessity of (45). We accordingly assume $U = \bar{U} = O$ so that the conjugate equations of (38') become

$$(47) \quad 2\pi O^n = \left\| \int_0^{2\pi} e^{-in\phi} d\bar{\tau}_{pq}(\phi) \right\|; \quad (n = 0, \pm 1, \dots).$$

On the other hand if we introduce a new integration variable θ in (38') by means of the transformation

$$(48) \quad \phi = 2\pi - \theta,$$

we obtain

$$(49) \quad -2\pi O^n = - \left\| \int_{2\pi}^0 e^{in(2\pi-\theta)} d\tau_{pq}(2\pi - \theta) \right\| = \left\| \int_0^{2\pi} e^{-in\theta} d\tau_{pq}(2\pi - \theta) \right\|. \\ (n = 0, \pm 1, \dots).$$

On adding (47) and (49) we have from (46)

$$(50) \quad \|0\| = \left\| \int_0^{2\pi} e^{-in\phi} d\omega_{pq}(\phi) \right\|; \quad (n = 0, \pm 1, \dots),$$

so that the Hermitian form $\omega(\phi; x) = \Phi[\|\omega_{pq}(\phi)\|; x]$ is, for any arbitrary given unit vector x , a solution of the homogeneous moment problem (28'). Furthermore the Hermitian form $\Phi[\|\tau_{pq}(\phi)\|; x]$ and consequently the Hermitian form $\Phi[\|\bar{\tau}_{pq}(\phi)\|; x]$ are, independently of the unit vector x , monotone non-decreasing functions of ϕ in the interval $[0, 2\pi]$. The Hermitian form $\Phi[\|\tau_{pq}(2\pi - \phi)\|; x]$ is, on the contrary, monotone non-increasing in $[0, 2\pi]$, for any given unit vector x . Accordingly the Hermitian form $\omega(\phi; x)$ is of bounded variation [cf. (46)] in the interval $[0, 2\pi]$ for any arbitrary given unit vector x . From the uniqueness property (33) of solutions of bounded variation of the homogeneous moment problem (28') we have, for an arbitrary given unit vector x ,

$$(51) \quad \omega(\phi; x) \equiv \Phi[\|\bar{\tau}_{pq}(\phi) + \tau_{pq}(2\pi - \phi)\|; x] = b = \text{const.},$$

at the continuity points of $\omega_{pq}(\phi)$ in $(0, 2\pi)$, and consequently, since (51) holds for an arbitrary unit vector x , the necessity of the condition (45) of I. Furthermore, since $\|\tau_{pq}(\phi)\|$ is Hermitian, the constant b in (51) is real.

The statement II is an immediate consequence of the necessary condition

(45) and the normalizing conditions (41'), (43'), (44'). From (45) we obtain on observing (41')

$$(52) \quad \bar{\tau}_{pq}(\phi) - \bar{\tau}_{pq}(0) = b\delta_{pq} - \tau_{pq}(2\pi - \phi)$$

at the continuity points of $\omega_{pq}(\phi)$ in $(0, 2\pi)$. Since the continuity points of $\omega_{pq}(\phi)$, which is obviously of bounded variation in $[0, 2\pi]$, are certainly everywhere dense in the interval $(0, 2\pi)$ we have for $\phi \rightarrow +0$ [cf. (30)] from (43') and (44')

$$(53) \quad \Delta_{0+0} \bar{\tau}_{pq}(\phi) = (b - 2\pi)\delta_{pq} \quad \text{where} \quad b = \bar{b} = \text{const.},$$

i. e. II is proven.

We now show the sufficiency of the condition (45) in I. From (38') for $n = 1$ we have as a consequence of (43')

$$\begin{aligned} 2\pi U = & \left\| \int_0^{2\pi} e^{i\phi} d\tau_{pq}(\phi) \right\| = \left\| \Delta_{0+0} \tau_{pq}(\phi) \right\| \\ & + \left\| \int_{0+0}^{\pi-0} e^{i\phi} d\tau_{pq}(\phi) \right\| - \left\| \Delta_{\pi} \tau_{pq}(\phi) \right\| + \left\| \int_{\pi+0}^{2\pi-0} e^{i\phi} d\tau_{pq}(\phi) \right\|, \end{aligned}$$

which may, on introducing a new integration variable θ by means of (48) in $\int_{\pi+0}^{2\pi-0}$, be written in the form

$$(54) \quad \begin{aligned} 2\pi U = & \left\| \Delta_{0+0} \tau_{pq}(\phi) \right\| + \left\| \int_{0+0}^{\pi-0} e^{i\phi} d\tau_{pq}(\phi) \right\| \\ & - \left\| \Delta_{\pi} \tau_{pq}(\phi) \right\| - \left\| \int_{0+0}^{\pi-0} e^{-i\theta} d\tau_{pq}(2\pi - \theta) \right\|. \end{aligned}$$

We now utilize (45) to obtain

$$(55) \quad \int_{0+0}^{\pi-0} e^{-i\phi} d\omega_{pq}(\phi) = \int_{0+0}^{\pi-0} e^{-i\phi} d\bar{\tau}_{pq}(\phi) + \int_{0+0}^{\pi-0} e^{-i\phi} d\tau_{pq}(2\pi - \phi) = 0,$$

inasmuch as the values of $\omega_{pq}(\phi)$ at the inner discontinuity points are immaterial in the Stieltjes integration. From (55) we may write (54) in the form

$$2\pi U = \left\| \Delta_{0+0} \tau_{pq}(\phi) \right\| + \left\| \int_{0+0}^{\pi-0} e^{i\phi} d\tau_{pq}(\phi) + \int_{0+0}^{\pi-0} e^{-i\phi} d\bar{\tau}_{pq}(\phi) \right\| - \left\| \Delta_{\pi} \tau_{pq}(\phi) \right\|,$$

so that from II there only remains to show that $\left\| \Delta_{\pi} \tau_{pq}(\phi) \right\|$ is a real matrix.

This follows immediately on writing (45) in the form

$$\tau_{pq}(2\pi - \phi) - \tau_{pq}(\phi) = b\delta_{pq} - [\tau_{pq}(\phi) + \bar{\tau}_{pq}(\phi)],$$

valid for the continuity points of $\omega_{pq}(\phi)$ in $(0, 2\pi)$, and then taking the limit for $\phi \rightarrow \pi - 0$.

II. APPLICATION OF THE TRIGONOMETRICAL MOMENT PROBLEM TO ORTHOGONAL MATRICES.

Among the properties of the spectral matrix $\| \tau_{pq}(\phi) \|$ of a given unitary matrix U , Wintner † has shown that it is possible to employ it in the construction of a bounded Hermitian matrix H

$$(56) \quad H = \| (1/2\pi) \int_0^{2\pi} \phi d\tau_{pq}(\phi) \|,$$

the elements of the spectral matrix $\| \sigma_{pq}(\phi) \|$ of which are given by

$$(57) \quad \begin{aligned} \sigma_{pq}(\phi) &= 0 && \text{for } -\infty < \phi \leq 0, \\ 2\pi\sigma_{pq}(\phi) &= \tau_{pq}(\phi) && \text{for } 0 \leq \phi \leq 2\pi, \\ \sigma_{pq}(\phi) &= \delta_{pq} && \text{for } 2\pi \leq \phi < +\infty. \end{aligned}$$

Conversely if H is a bounded Hermitian matrix whose spectral matrix $\| \sigma_{pq}(\phi) \|$ fulfills the conditions

$$(58) \quad \begin{aligned} \sigma_{pq}(\phi) &= 0 && \text{for } -\infty < \phi \leq 0, \\ \sigma_{pq}(2\pi - 0) &= \sigma_{pq}(\phi) = \delta_{pq} && \text{for } 2\pi \leq \phi < +\infty, \end{aligned}$$

(i. e. $\phi = 2\pi$ is not a point of the point spectrum of H) and if a matrix $\| \tau_{pq}(\phi) \|$ be introduced by means of the definitions

$$(59) \quad \tau_{pq}(\phi) = 2\pi\sigma_{pq}(\phi) \quad \text{for } 0 \leq \phi \leq 2\pi,$$

then the matrix

$$(60) \quad U = e^{iH} = \| (1/2\pi) \int_0^{2\pi} e^{i\phi} d\tau_{pq}(\phi) \|,$$

is a unitary matrix possessing as its spectral matrix the matrix $\| \tau_{pq}(\phi) \|$. If we designate a Hermitian matrix whose spectrum lies in the interval $[0, 2\pi]$ and whose point spectrum does not contain the point 2π , as a reduced Hermitian matrix we may sum up the above results in the following form: There exists between the unitary matrices and the reduced Hermitian matrices a one-to-one correspondence, i. e. to a given unitary matrix U there is associated a reduced Hermitian matrix H by means of (56) and to this same reduced Hermitian matrix H there is associated, by means of (60), the previously given unitary matrix U .

In the following we propose to apply the results of the previous section in order to characterize that sub-class of the reduced Hermitian matrices which is in one-to-one correspondence with the real unitary matrices, i. e.

† See "W", pp. 268-277.

the orthogonal matrices. In any case we may write any Hermitian matrix uniquely in the form

$$(61) \quad H = \pi B + iS,$$

where B is a real symmetric matrix and S a real skew-symmetric matrix so that our problem † is to characterize the matrix pair (B, S) for which the reduced Hermitian matrix $H = \pi B + iS$ has the property that the matrix $U = e^{iH}$ is real. As an answer to this problem we show

THEOREM I. *The necessary and sufficient conditions in order that the reduced Hermitian matrix $H = \pi B + iS$ yield a real unitary matrix O , i. e. an orthogonal matrix, in the one-to-one representation*

$$(62) \quad O = e^{iH} = e^{i(\pi B + iS)},$$

are that S be a bounded real skew-symmetric matrix and B a real symmetric idempotent matrix which is commutable with S , i. e.

$$(63a) \quad B = \bar{B} = B', \quad B^2 = B; \quad S = \bar{S} = -S',$$

$$(63b) \quad BS = SB.$$

Before beginning the demonstration of Theorem I we note that the representation (62) may, from (63b) and the distributive property (18) of the exponential with respect to commutable matrices, be given the form

$$(62') \quad O = e^{i\pi B} \cdot e^{-S} = e^{-S} \cdot e^{i\pi B}.$$

It may also be mentioned that the representation (62) or its equivalent (62') is "covariant" under orthogonal matrix transformation. More precisely if V is any orthogonal matrix then we have from (62)

$$VOV^{-1} = \exp [i\pi(VBV^{-1} + iVSV^{-1})],$$

where VBV^{-1} is a real symmetric idempotent matrix [cf. p. 585] commutable with the bounded real skew-symmetric matrix VSV^{-1} and $\pi VBV^{-1} + iVSV^{-1}$ is a reduced Hermitian matrix if $\pi B + iS$ is a reduced Hermitian matrix. This "covariance" property is important in what follows inasmuch as without it the classification of the orthogonal matrices in four classes as developed below would be meaningless.

We begin with a demonstration of the necessity of the conditions (63a) and (63b) and shall retain the notation $\|\tau_{pq}(\phi)\|$ for the spectral matrix of a unitary matrix U . First we show that,

† See "W", p. 277.

$$(64) \quad \int_{0+0}^{\pi-0} d\bar{\tau}_{pq}(\phi) = \int_{\pi+0}^{2\pi-0} d\tau_{pq}(\phi),$$

$$(65) \quad \int_{\pi+0}^{2\pi-0} \phi d\tau_{pq}(\phi) = 2\pi \int_{0+0}^{\pi-0} d\bar{\tau}_{pq}(\phi) - \int_{0+0}^{\pi-0} \phi d\bar{\tau}_{pq}(\phi).$$

On introducing a new integration variable θ by means of (48) we obtain for the left-hand members of (64) and (65) respectively

$$(66) \quad \int_{0+0}^{\pi-0} d\bar{\tau}_{pq}(\phi) = \int_{2\pi-0}^{\pi+0} d\bar{\tau}_{pq}(2\pi - \theta),$$

$$(67) \quad \int_{\pi+0}^{2\pi-0} \phi d\tau_{pq}(\phi) = \int_{\pi-0}^{0+0} (2\pi - \theta) d\tau_{pq}(2\pi - \theta).$$

From (45) we obtain

$$(68) \quad \int_{2\pi-0}^{\pi+0} d\omega_{pq}(\phi) = \int_{2\pi-0}^{\pi+0} d\bar{\tau}_{pq}(\phi) + \int_{2\pi-0}^{\pi+0} d\tau_{pq}(2\pi - \phi) = 0,$$

$$(69) \quad \int_{\pi-0}^{0+0} (2\pi - \phi) d\omega_{pq}(\phi) = \int_{\pi-0}^{0+0} (2\pi - \phi) d\bar{\tau}_{pq}(\phi) \\ + \int_{\pi-0}^{0+0} (2\pi - \phi) d\tau_{pq}(2\pi - \phi) = 0,$$

inasmuch as the values of $\omega_{pq}(\phi)$ at its inner discontinuity points play no rôle in the Stieltjes integration in (68) and (69). Formulas (64) and (65) obviously follow from (66) and (67) by virtue of (68) and (69), respectively. The reduced Hermitian matrix H in the exponential representation of Theorem I may always be calculated from the spectral matrix of the given unitary matrix U by means of (56). For the treatment of a real unitary matrix $U = \bar{U} = O$ we express the Stieltjes integration in (56) as follows:

$$(70) \quad 2\pi H = \left\| \int_{0+0}^{\pi-0} \phi d\tau_{pq}(\phi) \right\| + \pi \left\| \Delta_{\pi} \tau_{pq}(\phi) \right\| + \left\| \int_{\pi+0}^{2\pi-0} \phi d\tau_{pq}(\phi) \right\|,$$

inasmuch as [cf. (43') on p. 598]

$$\int_0^{0+0} \phi d\tau_{pq}(\phi) = \int_{2\pi-0}^{2\pi} \phi d\tau_{pq}(\phi) = 0.$$

By means of (65) we may rewrite (70) in the form

$$2\pi H = \left\| \int_{0+0}^{\pi-0} \phi d[\tau_{pq}(\phi) - \bar{\tau}_{pq}(\phi)] \right\| + \pi \left\| \Delta_{\pi} \tau_{pq}(\phi) \right\| + 2\pi \left\| \int_{0+0}^{\pi-0} d\bar{\tau}_{pq}(\phi) \right\|,$$

which in turn may be replaced by

$$(71) \quad 2\pi H = \parallel \int_{0+0}^{\pi-0} \phi d[\tau_{pq}(\phi) - \bar{\tau}_{pq}(\phi)] \parallel + \pi \parallel \int_{0+0}^{\pi-0} d[\bar{\tau}_{pq}(\phi) - \tau_{pq}(\phi)] \parallel \\ + \pi \parallel \int_{0+0}^{\pi-0} d\tau_{pq}(\phi) \parallel + \pi \parallel \Delta_{\pi} \tau_{pq}(\phi) \parallel + \pi \parallel \int_{0+0}^{\pi-0} d\bar{\tau}_{pq}(\phi) \parallel.$$

We introduce a bounded real skew-symmetric matrix S by means of

$$(72) \quad 2\pi S = -i \parallel \int_{0+0}^{\pi-0} \phi d[\tau_{pq}(\phi) - \bar{\tau}_{pq}(\phi)] \parallel - i\pi \parallel \int_{0+0}^{\pi-0} d[\bar{\tau}_{pq}(\phi) - \tau_{pq}(\phi)] \parallel \\ = -i \parallel \int_{0+0}^{\pi-0} (\phi - \pi) d[\tau_{pq}(\phi) - \bar{\tau}_{pq}(\phi)] \parallel,$$

and employ it, together with (64), to abbreviate (71) to

$$(73) \quad 2H = 2iS + \parallel \int_{0+0}^{\pi-0} d\tau_{pq}(\phi) \parallel + \parallel \Delta_{\pi} \tau_{pq}(\phi) \parallel + \parallel \int_{\pi+0}^{2\pi-0} d\tau_{pq}(\phi) \parallel.$$

Now from (38') [cf. p. 598] for $n=0$ we have

$$2\pi E = \parallel \int_0^{2\pi} d\tau_{pq}(\phi) \parallel = \parallel \Delta_{0+0} \tau_{pq}(\phi) \parallel \\ + \parallel \int_{0+0}^{\pi-0} d\tau_{pq}(\phi) \parallel + \parallel \Delta_{\pi} \tau_{pq}(\phi) \parallel + \parallel \int_{\pi+0}^{2\pi-0} d\tau_{pq}(\phi) \parallel,$$

inasmuch as from (43')

$$\int_{2\pi-0}^{2\pi} d\tau_{pq}(\phi) = 0,$$

so that (73) may be further shortened to

$$(74) \quad H = iS + \pi E - (1/2) \parallel \Delta_{0+0} \tau_{pq}(\phi) \parallel = iS + \pi[E - (1/2\pi) \parallel \Delta_{0+0} \tau_{pq}(\phi) \parallel],$$

in which S is the bounded real skew-symmetric matrix introduced in (72) and $\parallel \Delta_{0+0} \tau_{pq}(\phi) \parallel$, since $\parallel \tau_{pq}(\phi) \parallel$ is a bounded Hermitian matrix, is, from II on p. 599, a bounded real symmetric matrix. We introduce a bounded real [cf. (61)] symmetric matrix B by means of

$$B = E - (1/2\pi) \parallel \Delta_{0+0} \tau_{pq}(\phi) \parallel,$$

and which, from (59), may obviously be written in the form

$$(75) \quad B = E - \parallel \Delta_{0+0} \sigma_{pq}(\phi) \parallel,$$

so that, inasmuch as $\parallel \sigma_{pq}(\phi) \parallel$ is the spectral matrix of the reduced Hermitian matrix H we have †

† See "Hi", pp. 141-145. Cf. also "S", p. 160 and p. 174.

$$B^2 = E - 2 \left\| \Delta_{0+0} \sigma_{pq}(\phi) \right\| + \left\| \Delta_{0+0} \sigma_{pq}(\phi) \right\|^2 = E - \left\| \Delta_{0+0} \sigma_{pq}(\phi) \right\| = B,$$

i. e. B is a real symmetric idempotent matrix so that the necessity of (63a) is demonstrated.

In order to demonstrate the necessity of (63b) we write (72) in the form

$$(76) \quad 2\pi S = \lim_{n \rightarrow +\infty} 2\pi S_n,$$

where

$$2\pi S_n = -i \left\| \sum_{\nu=1}^{n-2} \left(\frac{\nu}{n} \pi - \pi \right) \left\{ \left[\tau_{pq} \left(\frac{\nu+1}{n} \pi \right) - \tau_{pq} \left(\frac{\nu}{n} \pi \right) \right] - \left[\bar{\tau}_{pq} \left(\frac{\nu+1}{n} \pi \right) - \bar{\tau}_{pq} \left(\frac{\nu}{n} \pi \right) \right] \right\} \right\|.$$

or, according to (59),

$$(77) \quad S_n = -i \left\| \sum_{\nu=1}^{n-2} \left(\frac{\nu}{n} \pi - \pi \right) \left\{ \left[\sigma_{pq} \left(\frac{\nu+1}{n} \pi \right) - \sigma_{pq} \left(\frac{\nu}{n} \pi \right) \right] - \left[\bar{\sigma}_{pq} \left(\frac{\nu+1}{n} \pi \right) - \bar{\sigma}_{pq} \left(\frac{\nu}{n} \pi \right) \right] \right\} \right\|.$$

We now apply the well-known theorem of Hilbert † that

$$\left\| \sigma_{pq}(\phi_1) \right\| \left\| \sigma_{pq}(\phi_2) \right\| = \left\| \sigma_{pq}(\phi_2) \right\| \left\| \sigma_{pq}(\phi_1) \right\|,$$

to obtain

$$(78) \quad \left\| \Delta_{0+0} \sigma_{pq}(\phi) \right\| \left\| \sigma_{pq}(\nu\pi/n) \right\| = \left\| \sigma_{pq}(\nu\pi/n) \right\| \left\| \Delta_{0+0} \sigma_{pq}(\phi) \right\|;$$

$$(\nu = 1, 2, \dots, n-1); (n = 2, 3, \dots),$$

and hence also

$$(79) \quad \left\| \Delta_{0+0} \sigma_{pq}(\phi) \right\| \left\| \bar{\sigma}_{pq}(\nu\pi/n) \right\| = \left\| \bar{\sigma}_{pq}(\nu\pi/n) \right\| \left\| \Delta_{0+0} \sigma_{pq}(\phi) \right\|;$$

$$(\nu = 1, 2, \dots, n-1); (n = 2, 3, \dots),$$

inasmuch as, from II on p. 599, the matrix $\left\| \Delta_{0+0} \sigma_{pq}(\phi) \right\|$ is a real matrix.

From (75), (77), (78) and (79) we accordingly have

$$(E - B)S_n = S_n(E - B),$$

from which and (76) then follows, for $n \rightarrow +\infty$, the necessity of (63b).

We now demonstrate the converse part of Theorem I, i. e. we assume that $H = \pi B + iS$ is a reduced Hermitian matrix for which the matrices

† See "Hi", pp. 141-145.

B and S fulfill the conditions (63a) and (63b) and demonstrate that U is a real matrix. We precede the proof by the derivation of the identity

$$(80) \quad e^{i\theta I} = E + (e^{i\theta} - 1)I,$$

where θ is an arbitrary real or complex number and I any bounded idempotent matrix (cf. p. 585). Since I is assumed to be bounded we have the convergent development (cf. p. 588)

$$e^{i\theta I} = \sum_{\nu=0}^{+\infty} \frac{(i\theta I)^\nu}{\nu!} = \sum_{\nu=0}^{+\infty} \frac{(i\theta I)^{2\nu}}{(2\nu)!} + \sum_{\nu=0}^{+\infty} \frac{(i\theta I)^{2\nu+1}}{(2\nu+1)!},$$

which may, since I is idempotent, obviously be written in the form

$$e^{i\theta I} = E - I + \left[\sum_{\nu=0}^{+\infty} \frac{(i\theta)^{2\nu}}{(2\nu)!} + \sum_{\nu=0}^{+\infty} \frac{(i\theta)^{2\nu+1}}{(2\nu+1)!} \right] I = E - I + \left[\sum_{\nu=0}^{+\infty} \frac{(i\theta)^\nu}{\nu!} \right] I,$$

from which (80) follows at once. In an analogous manner it is possible to obtain the identity

$$e^{i\theta Q} = E \cos \theta + iQ \sin \theta,$$

valid for arbitrary real or complex θ and for an arbitrary square root Q of the unit matrix E (cf. p. 585). From the commutability condition (63b) we obtain (62') from which if we apply (80) for $\theta = \pi$ and $I = B$ we obtain

$$e^{i\pi B} = (E - 2B) \cdot e^{-S} = e^{-S} \cdot (E - 2B),$$

which is, because of (63a), certainly a real matrix. q. e. d.

III. A CONNECTION BETWEEN REDUCED SKEW-SYMMETRIC AND REDUCED HERMITIAN MATRICES.

For convenience we adopt the notation

$$(82) \quad M \sim N$$

for the orthogonal equivalence of two matrices, i. e. for the existence of an orthogonal matrix O for which $M = ONO^{-1}$. We introduce the definition

A real skew-symmetric matrix S will be said to be reduced if the spectrum of the Hermitian matrix iS is contained in the closed interval $[-\pi, +\pi]$ and if the end-points $-\pi$ and $+\pi$ are not contained in the point spectrum of iS .

The property of a real skew-symmetric matrix to be reduced is an obvious orthogonal invariant.

We now show

LEMMA I. *Let B be a real symmetric idempotent matrix and let S be a real bounded skew-symmetric matrix commutable with B, i. e.*

$$(83) \quad BS = SB,$$

then the necessary and sufficient conditions in order that the Hermitian matrix $\pi B + iS$ be reduced (cf. p. 588) are

$$(84a) \quad (E - B)S = S(E - B) = \|0\|,$$

$$(84b) \quad S \text{ be reduced.}$$

Since B is a real symmetric idempotent matrix we have, according to a theorem of Hilbert,[†] in the notation (82)

$$(85) \quad B \sim \| \kappa_{pq} \| \text{ where } \kappa_{pq} = 0 \text{ for } p \neq q \text{ and } \kappa_{pp} = 0 \text{ or } \kappa_{pp} = +1.$$

We designate by α' and β' the number of times 0 and +1 respectively occur in the sequence $\{\kappa_{pp}\}$ so that necessarily $\alpha' \geq 0$, $\beta' \geq 0$ and $\alpha' + \beta' = +\infty$ and distinguish the following four possibilities for the matrix $\| \kappa_{pq} \|$ in (85):

$$(86) \quad \begin{array}{lll} \text{I:} & \alpha' = +\infty, & \beta' = 2m \quad \text{for } 0 \leq m < +\infty, \\ \text{II:} & \alpha' = +\infty, & \beta' = 2m + 1 \quad \text{for } 0 \leq m < +\infty, \\ \text{III:} & \alpha' = m, & \beta' = +\infty \quad \text{for } 0 \leq m < +\infty, \\ \text{IV:} & \alpha' = +\infty, & \beta' = +\infty, \end{array}$$

and corresponding to these four possibilities for the matrix $\| \kappa_{pq} \|$ in (85) we have the following four canonical forms for the matrix B:

$$(87) \quad \begin{array}{ll} \text{I:} & B \sim B^{(1)} = \| b^{(1)}_{pq} \|, \text{ where } b^{(1)}_{pq} = 0 \text{ for } p \neq q, \\ & \text{and } b^{(1)}_{pp} = 1 \text{ for } p = 1, 2, \dots, 2m < +\infty, \\ & \quad \quad \quad b^{(1)}_{pp} = 0 \text{ for } p > 2m, \\ \text{II:} & B \sim B^{(2)} = \| b^{(2)}_{pq} \|, \text{ where } b^{(2)}_{pq} = 0 \text{ for } p \neq q, \\ & \text{and } b^{(2)}_{pp} = 1 \text{ for } p = 1, 2, \dots, 2m + 1 < +\infty, \\ & \quad \quad \quad b^{(2)}_{pp} = 0 \text{ for } p > 2m + 1, \\ \text{III:} & B \sim B^{(3)} = \| b^{(3)}_{pq} \|, \text{ where } b^{(3)}_{pq} = 0 \text{ for } p \neq q, \\ & \text{and } b^{(3)}_{pp} = 0 \text{ for } p = 1, 2, \dots, m < +\infty, \\ & \quad \quad \quad b^{(3)}_{pp} = 1 \text{ for } p > m, \\ \text{IV:} & B \sim B^{(4)} = \| b^{(4)}_{pq} \|, \text{ where } b^{(4)}_{pq} = 0 \text{ for } p \neq q, \\ & \text{and } b^{(4)}_{pp} = b^{(4)}_{p+1p+1} = 1 \text{ for } p = 1, 5, 9, \dots, \\ & \quad \quad \quad b^{(4)}_{pp} = b^{(4)}_{p+1p+1} = 0 \text{ for } p = 3, 7, 11, \dots \end{array}$$

[†] See "Hi", p. 145.

Since the commutability property (83) assumed for the matrices B and S in Lemma I is invariant under simultaneous transformations of B and S we must have, simultaneously with the matrix transformation of the matrix B of Lemma I into each of the above canonical forms, the reduction of the skew-symmetric matrix S of Lemma I to one of the following types:

$$\begin{aligned}
 \text{I: } S &\sim S^{(1)} = \left\| \begin{array}{c|c} S_{2m} & \vdots \\ \hline \vdots & S_{\infty-2m} \end{array} \right\|, \\
 S_{2m} &= \bar{S}_{2m} = -S'_{2m}, \quad S_{\infty-2m} = \bar{S}_{\infty-2m} = -S'_{\infty-2m}, \\
 \text{II: } S &\sim S^{(2)} = \left\| \begin{array}{c|c} S_{2m+1} & \vdots \\ \hline \vdots & S_{\infty-(2m+1)} \end{array} \right\|, \\
 S_{2m+1} &= \bar{S}_{2m+1} = -S'_{2m+1}, \quad S_{\infty-(2m+1)} = \bar{S}_{\infty-(2m+1)} = -S'_{\infty-(2m+1)}, \\
 \text{III: } S &\sim S^{(3)} = \left\| \begin{array}{c|c} S_m & \vdots \\ \hline \vdots & S_{\infty-m} \end{array} \right\|, \\
 S_m &= \bar{S}_m = -S'_m, \quad S_{\infty-m} = \bar{S}_{\infty-m} = -S'_{\infty-m}, \\
 \text{IV: } S &\sim S^{(4)} = \left\| \begin{array}{cccc} S_{11} & Z & S_{13} & Z \\ Z & S_{22} & Z & S_{23} \\ S_{31} & Z & S_{33} & Z \\ Z & S_{32} & Z & S_{44} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{array} \right\|, \\
 S_{pq} &= \bar{S}_{pq} = -S'_{qp} = \left\| \begin{array}{cc} \alpha_{pq} & \beta_{pq} \\ \gamma_{pq} & \delta_{pq} \end{array} \right\|, \quad Z = \left\| \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right\|.
 \end{aligned}
 \tag{88}$$

in which the capital letters represent matrices and the vacant spaces of the remaining two "quadrants" of I, II and III are occupied entirely by zeros. For example, the matrix $S^{(1)} = \| s^{(1)}_{pq} \|$ of (88) is a bounded real skew-symmetric matrix for which

$$\begin{aligned}
 (89) \quad s^{(1)}_{pq} &= 0, \quad \text{for } p = 1, 2, \dots, 2m, \quad 0 \leq 2m < q; \\
 &\text{and for } 0 \leq 2m < p, \quad q = 1, 2, \dots, 2m.
 \end{aligned}$$

We now prove the necessity of the conditions (84a), (84b) in Lemma I. As mentioned in connection with Theorem I the property of a Hermitian matrix to be reduced is an orthogonal invariant. Consequently the Hermitian matrices

$$(90) \quad \pi B^{(j)} + iS^{(j)}; \quad (j = 1, 2, 3, 4),$$

constructed from (87) and (88), are all reduced. In order to show the necessity of the conditions (84a), (84b) of Lemma I we consider separately each of the four possible canonical forms for the matrix B of Lemma I as given in (87).

1. *Proof of the necessity of (84a) for matrices B of Class I.* From I of (88) we are enabled to write

$$(91) \quad S^{(1)} = S_a^{(1)} + S_b^{(1)},$$

where, in the schematic notations of (88), (89), we have placed

$$(92) \quad S_a^{(1)} = \left\| \begin{array}{c} S_{2m} \\ \vdots \\ \vdots \\ \vdots \end{array} \right\|, \quad S_b^{(1)} = \left\| \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ S_{\infty-2m} \end{array} \right\|.$$

so that the matrix pair $(S_a^{(1)}, S_b^{(1)})$ is separated (cf. p. 582). Since the Hermitian matrix $\pi B^{(1)} + iS^{(1)}$ is reduced [cf. (90) and adjoining] the domain $\Delta(\pi B^{(1)} + iS^{(1)})$ lies in the closed interval $[0, 2\pi]$ (cf. p. 582), i. e. from (91)

$$(93) \quad 0 \leq \Phi[(\pi B^{(1)} + iS^{(1)}); x] = \pi \Phi[B^{(1)}; x] \\ + \Phi[iS_a^{(1)}; x] + \Phi[iS_b^{(1)}; x] \leq 2\pi \quad \text{for } |x| = 1.$$

Since (93) holds for an arbitrary unit vector x we may choose in (93)

$$(94) \quad x_1 = x_2 = \cdots = x_{2m} = 0, \quad \sum_{v=2m+1}^{+\infty} |x_v|^2 = 1,$$

so that, from (92) and the definition of $B^{(1)}$ in (87), we derive from (93)

$$(95) \quad 0 \leq \Phi[iS_b^{(1)}; x] \leq 2\pi,$$

first, for only those unit vectors x of (94), and then, because of the peculiar definition of $S_b^{(1)}$ given in (92), for an arbitrary unit vector x . Since the domain of any bounded Hermitian matrix of the type iS , where S is a real skew-symmetric matrix, is necessarily symmetrical with respect to the origin (cf. p. 582) one must perforce replace (95) by

$$(96) \quad \Phi[iS_b^{(1)}; x] \equiv 0 \quad \text{for } |x| = 1,$$

and therefore from the uniqueness property of compound forms (cf. p. 580)

$$S_b^{(1)} = \|0\|,$$

or from (91)

$$(97) \quad S^{(1)} = S_a^{(1)}.$$

From (87), (97), and (92) it is clear that the matrix pair $(E - B^{(1)}, S^{(1)})$

is separated and consequently *a fortiori* perpendicular. Since the perpendicularity of a matrix pair is invariant with respect to simultaneous matrix transformations of the two matrices in the pair (cf. p. 583) the matrix pair $(E - B, S)$ of Lemma I is itself perpendicular and the necessity of (84a) is accordingly demonstrated, under the temporary restriction that B be of Class I.

2. *Proof of the necessity of (84b) for matrices B of Class I.* We choose instead of (94) those unit vectors x for which

$$(98) \quad \sum_{\nu=1}^{2m} |x_{\nu}|^2 = 1, \quad x_{2m+1} = x_{2m+2} = \cdots = 0,$$

so that, from (92) and the definition of $B^{(1)}$ in (87) we derive from (93)

$$(99) \quad 0 \leq \pi + \Phi[iS_a^{(1)}; x] \leq 2\pi, \quad \text{i. e.} \quad -\pi \leq \Phi[iS_a^{(1)}; x] \leq +\pi,$$

first, for only those unit vectors of (98), and then because of the peculiar definition of $S_a^{(1)}$ given in (92), for an arbitrary unit vector x . Since, as has been mentioned in the introduction (cf. p. 581), the spectrum of any bounded matrix is contained in the domain of the matrix it is clear from (99) that the spectrum of $iS_a^{(1)}$ is contained in the closed interval $[-\pi, +\pi]$. Furthermore the matrix $E - B^{(1)}$ is a diagonal matrix the elements of which are 0 and $+1$ and forms with $S^{(1)}$ the separated matrix pair $(E - B^{(1)}, S^{(1)})$ [cf. (87), (97), (92)] so that from p. 584 and the fact that the Hermitian matrix $\pi B^{(1)} + iS^{(1)}$ is reduced [cf. (90) and adjoining] the points $-\pi$ and $+\pi$ cannot be in the point spectrum of $iS^{(1)}$, i. e. $S^{(1)}$ is a reduced skew-symmetric matrix. Since the property of a skew-symmetric matrix to be reduced is an orthogonal invariant the skew-symmetric matrix S of Lemma I is itself reduced and the necessity of (84b) is, under the temporary restriction that B be of type I of (87), accordingly demonstrated.

3. *Proof of the necessity of (84a) and (84b) for the matrices B of Classes II, III, and IV.* That (84a) and (84b) are necessary if the matrix B in Lemma I is of type II of (87) follows at once on replacing $2m$ by $2m + 1$ throughout the preceding demonstrations. In order to treat type III of (87) we may write, analogously to (91) and (92), from (88)

$$(100) \quad S^{(3)} = S_a^{(3)} + S_b^{(3)},$$

where

$$(101) \quad S_a^{(3)} = \left\| \begin{array}{cccc} S_m & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \end{array} \right\|, \quad S_b^{(3)} = \left\| \begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & S_{\infty-m} \end{array} \right\|,$$

so that the matrix pair $(S_a^{(3)}, S_b^{(3)})$ is separated. Since the matrix $\pi B^{(3)} + iS^{(3)}$ is reduced [cf. (90) and adjoining] we accordingly have (cf. p. 582) from (100)

$$(102) \quad 0 \leq \Phi[(\pi B^{(3)} + iS^{(3)}); x] \\ = \pi \Phi[B^{(3)}; x] + \Phi[iS_a^{(3)}; x] + \Phi[iS_b^{(3)}; x] \leq 2\pi \text{ for } |x| = 1,$$

and hence

$$(103) \quad 0 \leq \pi + \Phi[iS_b^{(3)}; x] \leq 2\pi \text{ for } x_1 = x_2 = \cdots x_m = 0, \sum_{\nu=m+1}^{+\infty} |x_\nu|^2 = 1,$$

$$(104) \quad 0 \leq \Phi[iS_a^{(3)}; x] \leq 2\pi \text{ for } \sum_{\nu=1}^m |x_\nu|^2 = 1, \quad x_{m+1} = x_{m+2} = \cdots = 0,$$

so that one may deduce [cf. (95), (99) and adjoining] from (103) and (104) that the matrix pair $(E - B^{(3)}, S^{(3)})$ is separated and that the skew-symmetric matrix $S^{(3)}$ is reduced and then, as a consequence, that the matrix pair $(E - B, S)$ of Lemma I is perpendicular and that the skew-symmetric matrix S is reduced. In order to treat the remaining case in which the matrix B of Lemma I is of the type IV of (87) we write

$$(105) \quad S^{(4)} = S_a^{(4)} + S_b^{(4)},$$

where, in the notations employed in (88), we understand

$$(106) \quad S_a^{(4)} = \left\| \begin{array}{cccc} S_{11} & Z & S_{13} & Z \\ Z & Z & Z & Z \\ S_{31} & Z & S_{33} & Z \\ Z & Z & Z & Z \end{array} \right\|, \quad S_b^{(4)} = \left\| \begin{array}{cccc} Z & Z & Z & Z \\ Z & S_{22} & Z & S_{24} \\ Z & Z & Z & Z \\ Z & S_{42} & Z & S_{44} \end{array} \right\|,$$

so that the matrix pair $(S_a^{(4)}, S_b^{(4)})$ is separated. Since the Hermitian matrix $\pi B^{(4)} + iS^{(4)}$ is reduced [cf. (90) and adjoining] we accordingly have (cf. p. 582) from (105)

$$(107) \quad 0 \leq \Phi[(\pi B^{(4)} + iS^{(4)}); x] = \pi \Phi[B^{(4)}; x] \\ + \Phi[iS_a^{(4)}; x] + \Phi[iS_b^{(4)}; x] \leq 2\pi \text{ for } |x| = 1,$$

and since x is an arbitrary unit vector in (107) we may put

$$(108) \quad x_1 = x_2 = x_5 = x_6 = x_9 = x_{10} = \cdots = 0; \\ |x_3|^2 + |x_4|^2 + |x_7|^2 + |x_8|^2 + \cdots = 1,$$

so that from (106) and the definition of $B^{(4)}$ in (87) we may derive from (107)

$$(109) \quad 0 \leq \Phi [iS_b^{(4)}; x] \leq 2\pi,$$

first, for only those unit vectors x in (108), and then, because of the peculiar definition of $S_b^{(4)}$ given in (106), for an arbitrary unit vector x . A treatment of inequality (109) analogous to that given for (95) then yields

$$S_b^{(4)} = \| 0 \|,$$

or from (105)

$$(110) \quad S^{(4)} = S_a^{(4)}.$$

From (87), (110), and (106) it is clear that the matrix pair $(E - B^{(4)}, S^{(4)})$ is separated and consequently perpendicular. From this point on the proof of the necessity of (84a) for those matrices B of Lemma I of type IV in (87) is identical with the proof given in the treatment of type I and is accordingly not repeated.

In order to show the necessity of (84b) if the matrix B in Lemma I is of type IV in (87) we choose, instead of (108), those unit vectors x for which

$$(111) \quad |x_1|^2 + |x_2|^2 + |x_5|^2 + |x_6|^2 + |x_9|^2 + |x_{10}|^2 = \cdots = 1, \\ x_3 = x_4 = x_7 = x_8 = \cdots = 0$$

so that now, from (106) and the definition of $B^{(4)}$ in (87), the inequality (107) is replaced by

$$(112) \quad 0 \leq \pi + \Phi [iS_a^{(4)}; x] \leq 2\pi, \text{ i. e. } -\pi \leq \Phi [iS_a^{(4)}; x] \leq +\pi,$$

first, only for those unit vectors x in (111), and then, because of the peculiar definition of $S_a^{(4)}$ given in (106), for an arbitrary unit vector x . By employing the same argument used in the discussion of (99) we learn that the spectrum of $iS_a^{(4)}$ is contained in the closed interval $[-\pi, +\pi]$ and that neither of the end-points $-\pi$ and $+\pi$ are contained in the point spectrum of $iS_a^{(4)}$ (cf. p. 610). Finally it follows, just as in the treatment on p. 610, that the matrix S in Lemma I is a reduced skew-symmetric matrix.

4. *Sufficiency of the conditions (84a) and (84b) in Lemma I.* We now turn to the proof of the sufficiency of (84a) and (84b) in Lemma I. Since B is now by hypothesis a real symmetric idempotent matrix and S a real skew-symmetric matrix commutable with B [the commutability condition (83) is a necessary but not a sufficient condition for the perpendicularity of the matrix pair $(E - B, S)$], we may by *simultaneous* orthogonal matrix transformations obtain the canonical forms in (87) and (88) for the matrices B and S , respectively. As has been mentioned before, the assumptions in

(84a) and (84b) for the matrices B and S are orthogonal invariants and since the property of a Hermitian matrix to be reduced is likewise an orthogonal invariant it will be sufficient to prove the sufficiency for the Hermitian matrices in (90). We consider separately the four possibilities in (87) and begin with the treatment of type I. From the perpendicularity of the matrix pair $(E - B^{(1)}, S^{(1)})$ there follows from (87) and (88) that $S_b^{(1)} = \|0\|$ in (92), i. e. from (91), the matrix equation (97). Since $S^{(1)}$ is a reduced skew-symmetric matrix it follows from (97), (92) and (87) that $\pi B^{(1)} + iS^{(1)}$ is a reduced Hermitian matrix (cf. p. 584); from which the sufficiency of (84a) and (84b) in Lemma I follows [if B is of type I in (87)]. The treatment of the types II and III of (87) obviously proceeds entirely along the same lines and accordingly is not repeated here. The treatment of type IV of (87), while proceeding in the same general manner, is however more complicated and consequently is given in some detail. From the perpendicularity of the matrix pair $(E - B^{(4)}, S^{(4)})$ there follows from (87) and (88) that $S_b^{(4)} = \|0\|$ in (106), i. e. from (105), the matrix equation (110). Since $S^{(4)}$ is a reduced skew-symmetric matrix it follows from (110), (106), and (87) that first, the spectrum of $\pi B^{(4)} + iS^{(4)}$ is contained in the closed interval $[0, 2\pi]$ and second, from p. 584, that the end-point 2π is not contained in the point spectrum of $\pi B^{(4)} + iS^{(4)}$; from which the sufficiency of (84a) in Lemma I for the matrices B of the type IV in (87) follows. q. e. d.

III. THE PARAMETRIC REPRESENTATION OF THE MANIFOLD OF THE ORTHOGONAL MATRICES.

We first employ Theorem I and Lemma I for the demonstration of the following two theorems:

THEOREM II. *Every orthogonal matrix O may be represented in the form*

$$(113) \quad O = e^{i(\pi B + iS)} = e^{i\pi B} \cdot e^{-S} = e^{-S} \cdot e^{i\pi B},$$

in which B is a real symmetric idempotent matrix and S a real bounded skew-symmetric matrix commutable with B , i. e.

$$(114a) \quad B = \bar{B} = B', \quad B^2 = B; \quad S = \bar{S} = -S',$$

$$(114b) \quad BS = SB.$$

Furthermore this representation is possible and unique if the matrix pair

($E - B, S$) is perpendicular and if the skew-symmetric matrix S is reduced, i. e. if

$$(115a) \quad (E - B)S = S(E - B) = \| 0 \|,$$

$$(115b) \quad S \text{ is reduced.}$$

THEOREM III. Every orthogonal matrix O may be represented in the form

$$(116) \quad O = 2C - e^{-S},$$

in which C is a real symmetric idempotent matrix and S a bounded real skew-symmetric matrix such that the matrix pair (C, S) is perpendicular, i. e.

$$(117a) \quad C = \bar{C} = C', \quad C^2 = C; \quad S = \bar{S} = -S',$$

$$(117b) \quad CS = SC = \| 0 \|,$$

and this representation is unique if the skew-symmetric matrix S in (116) is reduced. The matrix C is in fact the matrix $E - B$ of Theorem II.

Theorem II is a trivial consequence of Theorem I and Lemma I in that for its demonstration it is only necessary to replace the condition in Theorem I that the Hermitian matrix $\pi B + iS$ be reduced by the equivalent conditions (84a) and (84b) of Lemma I. That the representation (113), (114a), (114b) always yields an orthogonal matrix, i. e. a real unitary matrix even without the conditions (115a), (115b) is trivial inasmuch as (cf. p. 606)

$$O = e^{i(\pi B + iS)} = e^{i\pi B} \cdot e^{-S} = (E - 2B)e^{-S} = \bar{O},$$

from which it is clear that the prescribing of the conditions (115a) and (115b) is responsible for the uniqueness of the representation (113).

Theorem III is now demonstrated as a consequence of Theorem II and the identity (19) on p. 588 valid for every perpendicular pair (M, N) of bounded matrices. In order to show this we write the representation (113) as follows:

$$(118) \quad O = \exp [i(\pi B + iS)] = \exp [i(\pi E - \pi(E - B) + iS)] \\ = -\exp i[-\pi(E - B) + iS].$$

If we now choose the matrices B and S to fulfill the conditions (114a), (114b), (115a), (115b) the representation (113), (118) provides a unique representation of any given orthogonal matrix O and which may, from (19), be written in the form

$$O = E - e^{-i\pi(E-B)} - e^{-S},$$

which in turn, from (80) [$\theta = -\pi$ and $I = E - B$], may be written as

$$(119) \quad O = 2(E - B) - e^{-S},$$

inasmuch as for $B^2 = B$, as is well known,

$$(E - B)^2 = E^2 - 2B + B^2 = E - B.$$

The matrix $E - B$ in (119) may accordingly be identified as the matrix C in Theorem III. Finally it is clear that the representation (116), (117a), (117b) always yields an orthogonal matrix, even though S is not a reduced skew-symmetric matrix, for

$$\begin{aligned} OO' &= O'O = [2(E - B) - e^{-S}] [2(E - B) - e^S] \\ &= 4(E - B)^2 - 2(E - B)e^S - 2e^{-S}(E - B) + E \\ &= 4(E - B) - 2(E - B) - 2(E - B) + E = E. \end{aligned}$$

It is clear from the above considerations that the only effect on the representation (116) of prescribing that the skew-symmetric matrix S be reduced is to insure the uniqueness of the representation.

We now show

LEMMA II. *The formula $Q = E - 2I$ yields a one-to-one correspondence between the idempotent matrices I and the matrices Q which are square roots of E . This one-to-one correspondence has the obvious property that the matrices I and Q in it are simultaneously real and simultaneously symmetric.*

Let I be a given arbitrary (not necessarily Hermitian) idempotent matrix, i. e. $I^2 = I$. The matrix $Q = E - 2I$ is then uniquely determined and satisfies the matrix equation

$$Q^2 = (E - 2I)^2 = E^2 - 4I + 4I^2 = E,$$

so that Q is a square root of E .

Let Q be an arbitrary (not necessarily Hermitian) square root of E , i. e. $Q^2 = E$. The matrix $I = \frac{1}{2}(E - Q)$ is then uniquely determined and satisfies the matrix equation

$$I^2 = (\frac{1}{4})(E - Q)^2 = (\frac{1}{4})(E^2 - 2Q + Q^2) = (\frac{1}{2})(E - Q) = I,$$

so that I is an idempotent matrix.

That the correspondence $Q = E - 2I$ is one to one is trivial.

As a consequence of Theorem I and Lemma II we show

THEOREM IV. *Every orthogonal matrix O may be represented in the form*

$$(120) \quad O = Qe^{-S} = e^{-S}Q,$$

in which Q is a real symmetric square root of E (and therefore itself an orthogonal matrix inasmuch as $QQ' = Q'Q = Q^2 = E$), S a bounded real skew-symmetric matrix, commutable with Q , i. e.

$$(121a) \quad Q = \bar{Q} = Q', \quad Q^2 = E; \quad S = \bar{S} = -S',$$

$$(121b) \quad QS = SQ,$$

and this correspondence between the orthogonal matrices O and the matrix pairs (Q, S) for which the Hermitian matrices $(\pi/2)(E - Q) + iS$ are reduced is one-to-one.

The matrix B in the representation (62) is, from (63a), a real symmetric idempotent matrix. According to the above lemma there exists a uniquely determined real symmetric square root Q of E for which (62) may be written in the form

$$(122) \quad O = \exp i[(\pi/2)(E - Q) + iS] = \exp [i(\pi/2)(E - Q)] \cdot \exp [-S].$$

The matrix $B = \frac{1}{2}(E - Q)$ is idempotent so that (122) may, from (80) [$\theta = \pi$ and $I = (\frac{1}{2})(E - Q)$], be written in the desired form (120). Finally from Lemma II and Theorem I the correspondence (120), (122) between the orthogonal matrices O and the matrix pairs (Q, S) of (121a), (121b) is one-to-one provided that the Hermitian matrix $\pi B + iS$, i. e. $(\pi/2)(E - Q) + iS$ is reduced. q. e. d.

V. THE FOUR CLASSES OF ORTHOGONAL MATRICES.

According to Lemma II if Q is a real symmetric square root of E there exists a uniquely determined real symmetric idempotent matrix B for which

$$(123) \quad Q = E - 2B$$

holds. Corresponding to the four canonical forms for the matrices B , under orthogonal matrix transformations, given in (87) we accordingly obtain the following four canonical forms for the matrices Q which are square roots of E :

$$\begin{aligned} \text{I: } Q \sim Q^{(1)} &= \|q^{(1)}_{jk}\|, \quad q^{(1)}_{jk} = 0 \quad j \neq k, \quad q^{(1)}_{jj} = -1 \\ &\quad \text{for } j = 1, 2, \dots, 2m < +\infty, \quad q^{(1)}_{jj} = +1 \text{ for } j > 2m, \\ \text{II: } Q \sim Q^{(2)} &= \|q^{(2)}_{jk}\|, \quad q^{(2)}_{jk} = 0 \quad j \neq k, \quad q^{(2)}_{jj} = -1 \\ &\quad \text{for } j = 1, 2, \dots, (2m+1) < +\infty, \quad q^{(2)}_{jj} = +1 \text{ for } j > 2m+1, \\ (124) \text{ III: } Q \sim Q^{(3)} &= \|q^{(3)}_{jk}\|, \quad q^{(3)}_{jk} = 0 \quad j \neq k, \quad q^{(3)}_{jj} = +1 \\ &\quad \text{for } j = 1, 2, \dots, m < +\infty, \quad q^{(3)}_{jj} = -1 \text{ for } j > m, \end{aligned}$$

$$\text{IV: } Q \sim Q^{(4)} = \| q^{(4)}_{jk} \|, \quad q^{(4)}_{j\bar{k}} = 0 \quad j \neq k, \quad q^{(4)}_{jj} = q_{j+1, j+1} = -1 \\ \text{for } j = 1, 5, 9, \dots, \quad q^{(4)}_{jj} = q^{(4)}_{j+1, j+1} = +1 \quad \text{for } j = 3, 7, 11, \dots,$$

On pages 607-612, we have shown that simultaneously with the transformation of the matrix B of Theorem I to a given canonical form (87) and hence simultaneously with the reduction of the matrix Q , defined by (123), to a canonical form in (124), the skew-symmetric matrix S of Theorem I is transformed into one of the four following forms:

$$\begin{aligned} \text{I: } S &\sim S^{(1)} = \left\| \begin{array}{c} S_{2m} \\ \vdots \\ - \\ \vdots \\ S_{\infty-m} \end{array} \right\|, \quad S_{2m} = \bar{S}_{2m} = -S'_{2m}, \\ \text{II: } S &\sim S^{(2)} = \left\| \begin{array}{c} S_{2m+1} \\ \vdots \\ - \\ \vdots \\ S_{\infty-m} \end{array} \right\|, \quad S_{2m+1} = \bar{S}_{2m+1} = -S'_{2m+1}, \\ \text{III: } S &\sim S^{(3)} = \left\| \begin{array}{c} \vdots \\ - \\ \vdots \\ S_{\infty-m} \end{array} \right\|, \quad S_{\infty-m} = \bar{S}_{\infty-m} = -S'_{\infty-m}, \\ \text{IV: } S &\sim S^{(4)} = \left\| \begin{array}{cccc} S_{11} & Z & S_{13} & Z \\ Z & Z & Z & Z \\ S_{31} & Z & S_{33} & Z \\ Z & Z & Z & Z \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{array} \right\|, \\ & S_{pq} = \bar{S}_{pq} = -S'_{qp} = \left\| \begin{array}{cc} \alpha_{pq} & \beta_{pq} \\ \gamma_{pq} & \delta_{pq} \end{array} \right\|, \quad Z = \left\| \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right\|. \end{aligned} \tag{125}$$

[cf. for example for I of (125) formulas (91), (92), and (97)]. Since, according to Theorem IV, we may express any given orthogonal matrix O uniquely in the form (120) it is possible to effect a division of the orthogonal matrices into four classes, namely according as

$$\begin{aligned} \text{Class I: } O &\sim Q^{(1)} e^{-S^{(1)}}, & \text{Class II: } O &\sim Q^{(2)} e^{-S^{(2)}}, \\ \text{Class III: } O &\sim Q^{(3)} e^{-S^{(3)}}, & \text{Class IV: } O &\sim Q^{(4)} e^{-S^{(4)}}, \end{aligned} \tag{126}$$

in which the matrices $Q^{(i)}$, $S^{(i)}$ ($i = 1, 2, 3, 4$) are defined in (124) and (125). As has been mentioned in connection with Theorem I this classification is possible because of the invariance of the spectra and the "co-variance" property there pointed out with respect to orthogonal matrix transformations.

We now confine our attention to orthogonal matrices of Classes I and II in (126) and obtain a further simplification of the canonical forms in

(126) for these two classes. We shall understand by a canonical l -rowed skew-symmetric matrix $\mathbf{F}_l$ ($l < +\infty$) the skew-symmetric matrix obtained by "inserting" in succession the real binary skew-symmetric matrices

$$(127) \quad \mathbf{S}_{pp} = \begin{vmatrix} 0 & s_p \\ -s_p & 0 \end{vmatrix}, \quad 1 \leq p \leq l/2 \quad (l \text{ even}); \quad 1 \leq p \leq (l-1)/2 \quad (l \text{ odd}),$$

$$-\infty < s_p < +\infty$$

in the diagonal positions of the l -rowed square zero matrix. For even l every row and column of $\mathbf{F}_l$ contains one and only one s_l , the same being true for odd l with the exception of the first row and column which contain only zeros. As is well known every finite real skew-symmetric matrix $\mathbf{S}$ is orthogonally equivalent to a canonical skew-symmetric matrix so that the points in the spectrum of $\mathbf{S}$ are then necessarily

$$(128) \quad \pm is_1, \pm is_2, \dots, \pm is_{l/2} \quad \text{or} \quad 0, \pm is_1, \pm is_2, \dots, is_{(l-1)/2},$$

according as l is even or odd respectively. In an analogous manner we shall understand by an infinite canonical skew-symmetric matrix $\mathbf{F}_\infty$ the skew-symmetric matrix obtained by inserting infinitely many of the binary skew-symmetric matrices (127) in the diagonal positions of the infinite zero matrix. Those bounded skew-symmetric matrices, possessing no continuous spectrum, are analogous to finite real skew-symmetric matrices in that they are orthogonally equivalent† to an infinite canonical skew-symmetric matrix $\mathbf{F}_\infty$. With the use of canonical skew-symmetric matrices, as defined above, the canonical forms for the matrices of Classes I and II as set forth in (126) may be further simplified to

$$(129) \quad \text{Class I: } \mathbf{O} \sim \mathbf{Q}^{(1)} e^{-\mathbf{S}^{(1)}} \sim \mathbf{Q}^{(1)} e^{\mathbf{F}_{2m}}, \quad \text{Class II: } \mathbf{O} \sim \mathbf{Q}^{(2)} e^{-\mathbf{S}^{(2)}} \sim \mathbf{Q}^{(2)} e^{\mathbf{F}_{2m+1}},$$

inasmuch as, as is clear from I and II of (126) that

$$\mathbf{Q}^{(1)} e^{-\mathbf{S}^{(1)}} = \mathbf{Q}^{(1)} e^{-\mathbf{S}_{2m}}, \quad \mathbf{Q}^{(2)} e^{-\mathbf{S}^{(2)}} = \mathbf{Q}^{(2)} e^{-\mathbf{S}_{2m+1}}.$$

By employing the identity‡

† See "Hi", p. 163, where the above fact is demonstrated for completely continuous skew-symmetric matrices the proof permitting an immediate generalization to bounded skew-symmetric matrices, not possessing continuous spectra.

‡ The identity (130) may be derived on applying the Cayley-Hamilton theorem to the definition of $\exp \mathbf{M}$ as given on p. 588. On placing

$$\mathbf{F}_\theta = \begin{vmatrix} 0 & \theta \\ -\theta & 0 \end{vmatrix},$$

the Cayley-Hamilton theorem yields the recursion formulas

$$(130) \quad \exp \begin{vmatrix} 0 & \theta \\ -\theta & 0 \end{vmatrix} = \begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix}$$

and if l is odd also

$$\exp \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & \theta \\ 0 & -\theta & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{vmatrix},$$

in which θ denotes an arbitrary real or complex number, and the finite canonical l -rowed rotation matrices $\mathbf{O}_l$ defined by

$$(131) \quad \mathbf{O}_l = e^{\mathbf{F}_l},$$

we are enabled to write (129) in the form

$$(132) \quad \text{Class I: } \mathbf{O} \sim \mathbf{Q}^{(1)} \mathbf{O}_{2m}, \quad \text{Class II: } \mathbf{O} \sim \mathbf{Q}^{(2)} \mathbf{O}_{2m+1},$$

and consequently replace the classification of the orthogonal matrices $\mathbf{O}$ as given in (126) by

$$(133) \quad \begin{aligned} \text{Class I: } & \mathbf{O} \sim \mathbf{Q}^{(1)} e^{-\mathbf{S}^{(1)}} \sim \mathbf{Q}^{(1)} \mathbf{O}_{2m}, \\ \text{Class II: } & \mathbf{O} \sim \mathbf{Q}^{(2)} e^{-\mathbf{S}^{(2)}} \sim \mathbf{Q}^{(2)} \mathbf{O}_{2m+1}, \\ \text{Class III: } & \mathbf{O} \sim \mathbf{Q}^{(3)} e^{-\mathbf{S}^{(3)}}, \\ \text{Class IV: } & \mathbf{O} \sim \mathbf{Q}^{(4)} e^{-\mathbf{S}^{(4)}}. \end{aligned}$$

For finite orthogonal matrices there exist no classes which would correspond to Classes III and IV of (133) whereas Classes I and II correspond to the finite orthogonal matrices of determinants $+1$ and -1 respectively since, denoting by $\det \mathbf{M}$ the determinant of an arbitrary finite matrix $\mathbf{M}$, we have from † (124)

$$\mathbf{F}_\theta^{2j} = (-1)^j \theta^{2j} \mathbf{E}, \quad \mathbf{F}_\theta^{2j+1} = (-1)^j \theta^{2j} \mathbf{F}_\theta; \quad (j=0, 1, 2, \dots),$$

from which the matrix

$$e^{\mathbf{F}_\theta} = \sum_{\nu=0}^{+\infty} \frac{\mathbf{F}_\theta^\nu}{\nu!} = \sum_{\nu=0}^{+\infty} \frac{\mathbf{F}_\theta^{2\nu}}{(2\nu)!} + \sum_{\nu=0}^{+\infty} \frac{\mathbf{F}_\theta^{2\nu+1}}{(2\nu+1)!},$$

takes the form

$$\begin{aligned} e^{\mathbf{F}_\theta} &= \mathbf{E} \sum_{\nu=0}^{+\infty} (-1)^\nu \frac{\theta^{2\nu}}{(2\nu)!} + \frac{\mathbf{F}_\theta}{\theta} \sum_{\nu=0}^{+\infty} (-1)^\nu \frac{\theta^{2\nu+1}}{(2\nu+1)!} \\ &= \mathbf{E} \cos \theta + \frac{\sin \theta}{\theta} \mathbf{F}_\theta = \begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix}. \end{aligned}$$

Cf. the paper of Galanti referred to on p. 594.

† The first of the equations in (134) follows from the invariance of a determinant of a matrix with respect to matrix transformations, i.e.

$$\det e^{-\mathbf{S}} = \det \mathbf{O} e^{-\mathbf{S}} \mathbf{O}^{-1} = \det e^{-\mathbf{O} \mathbf{S} \mathbf{O}^{-1}},$$

and the easily verified fact that the finite canonical rotation matrices defined in (131) above are of determinant $+1$.

$$\begin{aligned}
 (134) \quad & \det e^{-\mathbf{S}} = (+1), \quad \mathbf{S} = \bar{\mathbf{S}} = -\mathbf{S}', \\
 & \det \mathbf{Q}^{(1)} = (-1)^{2m} = +1, \\
 & \det \mathbf{Q}^{(2)} = (-1)^{2m+1} = -1.
 \end{aligned}$$

VI. ROTATIONS AND REFLECTIONS.

We now return to infinite orthogonal matrices and propose the definition: *We designate an orthogonal matrix $\mathbf{O}$ as a rotation matrix if and only if there exists a bounded real skew-symmetric matrix $\mathbf{G}$ (not necessarily unique) for which*

$$(135) \quad \mathbf{O} = e^{\mathbf{G}},$$

holds. An orthogonal matrix not permitting this representation we designate as a reflection matrix.

In order to justify the introduction of this definition we show † the following lemma pertaining to finite orthogonal matrices:

LEMMA III. *Every orthogonal matrix of determinant +1 (i. e. a finite rotation matrix) permits the representation (135) and no orthogonal matrix of determinant -1 (i. e. a finite reflection matrix) permits this representation.*

Let $\mathbf{O}$ be a given orthogonal matrix of determinant +1, i. e. the matrix $\mathbf{O}$ is of Class I in (133) so that we have

$$(136) \quad \mathbf{O} \sim \mathbf{Q}^{(1)} e^{\mathbf{F}_{2m}},$$

in which the matrices $\mathbf{Q}^{(1)}$ and $\mathbf{F}_{2m}$ are defined on p. 616 and p. 618, respectively. If we now introduce a canonical skew-symmetric matrix $\mathbf{F}^{(1)}_{2m}$ defined by placing

$$\left\| \begin{array}{ccc} \frac{\pi^2 - 2b^2}{\pi^2} & -\frac{2b(\pi^2 - b^2)^{1/2}}{\pi^2} & 0 \\ -\frac{2b(\pi^2 - b^2)^{1/2}}{\pi^2} & \frac{\pi^2 - 2b^2}{\pi^2} & 0 \\ 0 & 0 & -1 \end{array} \right\| = \exp \left\| \begin{array}{ccc} 0 & 0 & b \\ 0 & 0 & (\pi^2 - b^2)^{1/2} \\ -b & -(\pi^2 - b^2)^{1/2} & 0 \end{array} \right\|$$

of an orthogonal matrix of determinant +1 which is already in canonical form and whose exponential representation (135) is not at all obvious from the identity (130).

$$s_i = 1 \quad (i = 1, 2, \dots),$$

in (128) we are enabled to write the matrix $\mathbf{Q}^{(1)}$ of (136), (124) in the exponential form

$$(137) \quad \mathbf{Q}^{(1)} = e^{\pi \mathbf{F}^{(1)}_{2m}},$$

inasmuch as, as follows from (130) for $\theta = \pi$, we have

$$\exp \pi \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}.$$

Since the matrices $\mathbf{F}_{2m}$ and $\mathbf{F}^{(1)}_{2m}$ are obviously commutable we may employ the distributive property of the exponential representation with respect to commutable matrices and write (136) with the help of (137) in the form

$$\mathbf{O} \sim e^{\pi \mathbf{F}^{(1)}_{2m}} \cdot e^{\mathbf{F}_{2m}} = \exp [\pi \mathbf{F}^{(1)}_{2m} + \mathbf{F}_{2m}],$$

and hence, since the property of being a rotation matrix in the sense of our definition is for finite as well as for infinite matrices obviously an orthogonal invariant,[†] the demonstration of the first part of Lemma III is completed.

The demonstration of the second part of Lemma III, namely that every finite matrix permitting the representation (133) is necessarily of determinant $+1$ is contained in (134) (cf. footnote thereto on page 619).

As an application of the exponential definition of rotation and reflection matrices as given in (135) we show

THEOREM V. *Every orthogonal matrix of Class I is, in the sense of the definition (135), a rotation matrix and every orthogonal matrix of Class II is, in the sense of definition (135), a reflection matrix.*

The demonstration of the first part of this theorem is seen to be identical, from the formula (136) on, of the first part of Lemma III and consequently is not repeated. As has been mentioned above, the property of an orthogonal matrix to be a rotation or a reflection matrix, in the sense of the definition on p. 620, is an orthogonal invariant. Consequently in order to demonstrate that the matrices of Class II in (133) are all reflection matrices it will be sufficient to demonstrate that the matrices $\mathbf{Q}^{(2)}\mathbf{O}_{2m+1}$ of II in (133) are all reflection matrices. The second part of Theorem V may not, as appears at

[†] This invariance property is trivial inasmuch as follows from the definition of $e\mathbf{G}$ on p. 588 that if $\mathbf{O}$ is an arbitrary orthogonal matrix then $\mathbf{O}e\mathbf{G}\mathbf{O}^{-1} = e\mathbf{G}\mathbf{O}\mathbf{O}^{-1}$ in which $\mathbf{O}\mathbf{G}\mathbf{O}^{-1}$ is again a bounded real skew-symmetric matrix.

first sight, be reduced to the fact that no finite orthogonal matrix of determinant -1 permits the representation (135) [cf. (130)], since it would be possible that $Q^{(2)}e^{F_{2m+1}} = Q^{(2)}O_{2m+1} = e^G$ in which G is a bounded real skew-symmetric matrix containing infinitely many elements different from zero. In order to give a correct demonstration we first show

LEMMA IV. *The spectrum of an arbitrary real bounded Hermitian matrix H (not necessarily reduced) is mapped upon the spectrum of the matrix e^{iH} by the transformation $w = e^{iz}$, i. e. if θ (θ real) is a point in the spectrum of H then $e^{i\theta}$ is a point in the spectrum of e^{iH} .*

The proof of this lemma is trivial for reduced Hermitian matrices as it follows directly from the connection between the spectral matrices of H and e^{iH} as given in (59). We shall however require the result of this lemma for an arbitrary Hermitian matrix H and accordingly give a direct proof valid even if the matrix is not reduced. We adopt the notations for an arbitrary Hermitian matrix M

$$(138) \quad \begin{array}{ll} l[M] & \text{greatest lower bound of } \Phi[M; x] \quad \text{for } |x| = 1, \\ u[M] & \text{least upper bound of } |\Phi[M; x]| \quad \text{for } |x| = 1. \end{array}$$

We shall require the well known inequalities †

$$(139) \quad |\Phi[MN; x]| \leq (\Phi[MM^*; x] \Phi[N^*N; x])^{1/2} \quad \text{for } |x| = 1,$$

$$(140) \quad u[M, N] \leq u[M] u[N],$$

$$(141) \quad l[MM^*] \geq 0 \quad l[M^*M] \geq 0,$$

valid for any two bounded matrices M and N .

First we derive the relations

$$(142) \quad \begin{aligned} \Phi[(\theta E - H)(\theta E - H)^*; x] \\ = \Phi[(\theta E - H)^*(\theta E - H); x] = \Phi[(\theta E - H)^2; x] \geq 0, \end{aligned}$$

$$(143) \quad \begin{aligned} \Phi[(e^{i\theta}E - e^{iH})(e^{i\theta}E - e^{iH})^*; x] \\ = \Phi[(e^{i\theta}E - e^{iH})^*(e^{i\theta}E - e^{iH}); x] = -2 \sum_{\nu=1}^{\infty} (-1)^{\nu} \frac{\Phi[(\theta E - H)^{2\nu}; x]}{(2\nu)!} \end{aligned}$$

$$(144) \quad 0 \leq \Phi[(\theta E - H)^{2k}; x] \leq (\Phi[(\theta E - H)^2; x] \Phi[(\theta E - H)^{4k-2}; x])^{1/2};$$

($k = 1, 2, \dots$),

valid for real θ , an arbitrary unit vector x , and an arbitrary bounded Hermitian matrix H . The proof of (142) is, from (141), trivial. For the demonstration of (143) we observe that from

† Cf. for instance "S", p. 137, p. 130, p. 132, respectively.

$$\begin{aligned}
 (145) \quad \Phi[(e^{i\theta}E - e^{iH})(e^{i\theta}E - e^{iH})^*; x] \\
 = \Phi[(e^{i\theta}E - e^{iH})(e^{-i\theta}E - e^{-iH}); x] \\
 = \Phi[(e^{i\theta}E - e^{iH})^*(e^{i\theta}E - e^{iH}); x],
 \end{aligned}$$

we obtain

$$(146) \quad \Phi[(e^{i\theta}E - e^{iH})(e^{i\theta}E - e^{iH})^*; x] = 2 - \Phi[(e^{i(\theta E - H)} + e^{-i(\theta E - H)}); x],$$

from which (143) obviously follows. The inequalities (144) are derived from that of (139) by placing therein $M = (\theta E - H)$, $N = (\theta E - H)^{2k-1}$ and $k = 2, 3, \dots$ in succession and at the same time employing (142) and

$$0 \leq \Phi[(\theta E - H)^2(\theta E - H)^{2*}; x] = \Phi[(\theta E - H)^4; x].$$

If H is a diagonal matrix the lemma is trivial and we exclude this case in what follows. Let θ be a point in the spectrum of H , i. e. assume [cf. (581)]

$$\begin{aligned}
 (147) \quad I[(\theta E - H)(\theta E - H)^*] \\
 = I[(\theta E - H)^*(\theta E - H)] = I[(\theta E - H)^2] = 0,
 \end{aligned}$$

so that there exists a sequence of unit vectors $\{x_n\}$ for which

$$(148) \quad \lim_{n \rightarrow +\infty} \Phi[(\theta E - H)^2; x_n] = 0.$$

From (142), (143), and (144) we have

$$\begin{aligned}
 (149) \quad 0 \leq \Phi[(e^{i\theta}E - e^{iH})(e^{i\theta}E - e^{iH})^*; x] &\leq 2 \sum_{\nu=1}^{+\infty} \frac{\Phi[(\theta E - H)^{2\nu}; x]}{(2\nu)!} \\
 &\leq 2(\Phi[(\theta E - H)^2; x])^{1/2} \sum_{\nu=1}^{+\infty} \frac{(\Phi[(\theta E - H)^{4\nu-2}; x])^{1/2}}{(2\nu)!}.
 \end{aligned}$$

Since $\theta E - H \neq 0$ is bounded we have

$$(150) \quad 0 < \mathbf{u}[\theta E - H] = \mathbf{u},$$

where we write, for the sake of brevity, $\mathbf{u}[\theta E - H] = \mathbf{u}$ and hence have from (138) and (140)

$$(\Phi[(\theta E - H)^{4\nu-2}; x])^{1/2} \leq (\mathbf{u}[(\theta E - H)^{4\nu-2}])^{1/2} \leq \mathbf{u}^{2\nu-1}; \quad (\nu = 1, 2, \dots),$$

so that the inequality (149) may be replaced by

$$\begin{aligned}
 (151) \quad 0 \leq \Phi[(e^{i\theta}E - e^{iH})(e^{i\theta}E - e^{iH})^*; x] \\
 \leq \frac{2}{\mathbf{u}} (\Phi[(\theta E - H)^2; x])^{1/2} \sum_{\nu=1}^{+\infty} \frac{\mathbf{u}^{2\nu}}{(2\nu)!} < \frac{2e^{\mathbf{u}}}{\mathbf{u}} (\Phi[(\theta E - H)^2; x])^{1/2}.
 \end{aligned}$$

If one now applies the inequality (151) to the sequence of vectors $\{x_n\}$ for which (148) holds there follows from (151) the fulfillment of (3) for

$$\lambda = e^{i\theta}, \quad M = e^{iH},$$

i. e. $e^{i\theta}$ is a point in the spectrum of e^{iH} .

If S is a real bounded skew-symmetric matrix the matrix iS is a bounded Hermitian matrix so that we have the

COROLLARY. *The spectrum of an arbitrary real bounded skew-symmetric matrix S is mapped upon the spectrum of the matrix e^S by the transformation $w = e^z$, i. e. if $i\theta$ is a point in the spectrum of S then $e^{i\theta}$ is a point in the spectrum of e^S .*

We now demonstrate the second part of Theorem V and first calculate the matrix $Q^{(2)}O_{2m+1}$ of (133) obtaining from (124), (132), (127) and (131)

$$(152) \quad Q^{(2)}O_{2m+1}$$

$$= \begin{vmatrix} -1 & & & & & & & \\ & -\cos s_1 & -\sin s_1 & & & & & \\ & \sin s_1 & -\cos s_1 & & & & & \\ & & & -\cos s_2 & -\sin s_2 & & & \\ & & & \sin s_2 & -\cos s_2 & & & \\ & & & & & \ddots & & \\ & & & & & & -\cos s_m & -\sin s_m \\ & & & & & & \sin s_m & -\cos s_m \\ & & & & & & & 1 \\ & & & & & & & & 1 \\ & & & & & & & & & \ddots \end{vmatrix},$$

from which it is clear that the matrix (152) possesses only a point spectrum namely the points

$$(153) \quad -1, e^{\pm i s_1}, e^{\pm i s_2}, \dots, e^{\pm i s_m}, 1, 1, \dots$$

so that -1 necessarily occurs an odd number of times in the spectrum of (152). If now the matrix (152) were to permit the exponential representation e^G of (135) the matrix G therein could, from the above corollary, possess no continuous spectra and consequently (cf. p. 618) is orthogonally equivalent to an infinite canonical skew-symmetric matrix F_∞ . Since the spectrum of any matrix is invariant under matrix transformation it is clear that either the points

$$(154) \quad (2k+1)\pi i, \quad -(2k+1)\pi i; \quad (k=0, 1, \dots),$$

belong simultaneously to the spectrum of G or neither belongs to the spectrum

of G . Therefore if -1 occurs in the spectrum of e^G it follows from (154) and the above corollary that it must occur an even or an infinite number of times which is a contradiction to (153). q. e. d.

APPENDIX.

THE PROBLEM OF MOMENTS FOR A FINITE INTERVAL AND THE THEORY OF BOUNDED HERMITIAN MATRICES.

Several presentations have been given for the spectral representation of a bounded Hermitian matrix by means of Stieltjes integrals as introduced by Hilbert. The various modes of investigation require in common with the method of Hilbert, as far as the author knows from the literature,† the use of the characteristic constants of the truncated matrices of the given bounded Hermitian matrix. This procedure has certain methodical inconveniences. First, the spectra of the truncated matrices of a bounded Hermitian matrix do not necessarily determine the spectrum of the infinite matrix itself. An example of such a bounded Hermitian matrix has been pointed out by Toeplitz‡ in which he considers the bounded Hermitian matrix constructed from the infinite zero matrix by inserting the real binary symmetric matrix

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

in the diagonal positions of the infinite zero matrix, i. e. the matrix of the Hermitian form

$$x_1\bar{x}_2 + x_2\bar{x}_1 + x_3\bar{x}_4 + x_4\bar{x}_3 + \dots,$$

and which contains only the points -1 and $+1$ in its spectrum whereas not only -1 and $+1$ but also 0 belong to the spectrum of infinitely many truncated matrices. Second, the method of the truncated matrices is restricted to the treatment of Hermitian matrices alone. For instance the truncated matrices of a unitary matrix are not necessarily also unitary matrices although it is possible to obtain by a direct treatment of the trigonometrical momentum problem belonging to the infinite matrix itself a spectral

† I. See "Hi", pp. 131-137; II. Hellinger Toeplitz, "Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten," *Encyklopädie der Mathematische Wissenschaften* 2III2, pp. 1575-1584. III. F. Riesz, "Über quadratische Formen von unendlichvielen Veränderlichen," *Göttingen Nachrichten* (1910), pp. 190-195. Cf. also by the same author "R", pp. 128-139. We refer to the article of Hellinger Toeplitz as "HT" and this paper of Riesz as "U".

‡ See "T", p. 107.

representation which is entirely analogous to that discovered by Hilbert † for bounded Hermitian matrices. It is shown in the present note that such a direct treatment is also possible for bounded Hermitian matrices and yields a very simple demonstration of the fundamental theorem of Hilbert. It is necessary to start, not with the problem of trigonometrical moments but with the ordinary problem of moments for a finite interval as developed by Hausdorff.‡ The method possesses, in contrast to the methods of F. Riesz, not only the advantage of avoiding the use § of the truncated matrices at all but also enables us to start directly with a monotone Stieltjes kernel (Belegungsfunktion) whereas F. Riesz deduces ¶ the result from his general theorem || on the solvability of a system of Stieltjes integral equations with a function of bounded variation and then has to prove that the solution is actually monotone. It is of course a matter of fact that the demonstration of the theorem of Hausdorff is based on the theorems of Helly applied by Carleman in connection with the method of the truncated matrices as developed by Hilbert. In addition to the theorem of Hausdorff we need the fact that every bounded positive definite matrix possesses a bounded "square root". If one is already in possession of the spectral representation of a bounded Hermitian matrix (the deduction of which is the purpose of this note) the uniqueness,†† as well as the existence of a bounded positive definite "square root" of a bounded positive definite matrix, is assured. We need, however, only the existence of a "square root" of a bounded positive definite matrix and this is available, according to a remark of eHillinger-Toeplitz,‡‡ without the use of a spectral representation.

For the sake of completeness we first discuss in some detail the remark of Hellinger-Toeplitz concerning the construction of a bounded positive definite square root of a bounded positive definite matrix. The basis of this remark is the ordinary Taylor development

† See "Hi", p. 138.

‡ See "Ha".

§ See "R", pp. 129-139, where Riesz gives a direct proof employing an approximation by matrix polynomials which however also depends upon the use of the truncated matrices.

¶ See "U".

|| F. Riesz, "Sur les opérations fonctionelles linéaires," *Comptes Rendus*, 29 novembre (1909), and "Sur certains systèmes d'équations fonctionelles et l'approximation des fonctions continues," *Comptes Rendus*, 14 mars (1910).

†† See "W", p. 267. For finite matrices a direct demonstration of this existence and uniqueness property is given by L. Autonne in "Sur l'Hermitien," *Rendiconti del Circolo Matematico di Palermo*, Vol. 16 (1902), pp. 120-121.

‡‡ See "HT", p. 1567, footnote (522 a).

$$(155) \quad f(z) = (1 - z)^{1/2} = \sum_{\nu=0}^{+\infty} a_{\nu} z^{\nu}, \quad |z| < 1,$$

in which

$$(156) \quad a_0 a_1 = 1, \quad a_0 a_1 + a_1 a_0 = -1, \quad \sum_{\mu=0}^n a_{\mu} a_{n-\mu} = 0; \quad (n = 2, 3, \dots).$$

We denote by

$$(157) \quad \Psi[A; x, y] = \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} a_{pq} x_p y_q,$$

the bilinear form of an arbitrary bounded matrix A and by

$$(158) \quad \Phi[A; x] = \Psi[A; x, \bar{x}] = \sum_{p=1}^{+\infty} \sum_{q=1}^{+\infty} a_{pq} x_p \bar{x}_q,$$

the compound form (Kopplungsform) of A . The upper limit of $|\Psi[A; x, y]|$ for those vectors x and y for which

$$(159) \quad |x| = \left(\sum_{\nu=1}^{+\infty} |x_{\nu}|^2 \right)^{1/2} = 1, \quad |y| = \left(\sum_{\nu=1}^{+\infty} |y_{\nu}|^2 \right)^{1/2} = 1,$$

will be denoted by $R(A)$, the upper and lower limits of $|\Phi[A; x]|$ for those vectors x of (159) being denoted by $M(A)$ and $m(A)$ respectively. It is known that if A_1 and A_2 are two arbitrary bounded matrices then according to the inequality of Schwarz †

$$(160) \quad R(A_1 A_2) \leq R(A_1) R(A_2).$$

Furthermore it is clear that the series

$$(161) \quad \sum_{\nu=0}^{+\infty} A_{\nu}$$

converges to a bounded matrix A if

$$(162) \quad \sum_{\nu=0}^{+\infty} R(A_{\nu}) < +\infty.$$

For an arbitrary bounded Hermitian matrix H it is known that the compound form $\Phi[H; x]$ takes only real values and that

$$(163) \quad R(H) = M(H).$$

A bounded Hermitian matrix for which the upper and lower limits of its compound form on the complex Hilbert sphere $|x| = 1$ of (159) are con-

† Cf. for instance "S", p. 130.

tained in the open interval $(0, 1)$ will be denoted by $\mathbf{K}$. We accordingly have for every Hermitian matrix $\mathbf{K}$

$$(164) \quad 0 < \mathbf{m}(\mathbf{K}) \leq \mathbf{M}(\mathbf{K}) < 1,$$

in which the equality sign holds for and only for those matrices $\gamma\mathbf{E}$ in which $\mathbf{E}$ is the unit matrix and $0 < \gamma < 1$. The matrix $\mathbf{E} - \mathbf{K}$ is likewise a bounded Hermitian matrix for which we have from (164)

$$(165) \quad 0 < \mathbf{m}(\mathbf{E} - \mathbf{K}) \leq \mathbf{M}(\mathbf{E} - \mathbf{K}) < 1.$$

The matrix

$$(166) \quad \mathbf{A} = \sum_{\nu=0}^{+\infty} a_{\nu} \mathbf{K}^{\nu},$$

in which the a_{ν} are the coefficients in the Taylor development (155) and consequently fulfill the bilinear relations (156), is a bounded matrix since, from (164) and the definition of the a_{ν} in (155), (156) we have from (160) and (163)

$$\sum_{\nu=0}^{+\infty} \mathbf{R}(a_{\nu} \mathbf{K}^{\nu}) = \sum_{\nu=0}^{+\infty} |a_{\nu}| \mathbf{R}(\mathbf{K}^{\nu}) \leq \sum_{\nu=0}^{+\infty} |a_{\nu}| \mathbf{R}(\mathbf{K})^{\nu} = \sum_{\nu=0}^{+\infty} |a_{\nu}| \mathbf{M}(\mathbf{K})^{\nu} < +\infty,$$

and therefore

$$\mathbf{R}(\mathbf{A}) \leq \sum_{\nu=0}^{+\infty} \mathbf{R}(a_{\nu} \mathbf{K}^{\nu}) < +\infty.$$

One then verifies easily from (166) and the definition of the a_{ν} in (155), (156) that

$$(167) \quad \mathbf{A}^2 = \mathbf{E} - \mathbf{K},$$

and, in an analogous manner from (165), that the matrix

$$(168) \quad \mathbf{B} = \sum_{\nu=0}^{+\infty} a_{\nu} (\mathbf{E} - \mathbf{K})^{\nu},$$

is bounded and fulfills the matrix equation

$$(169) \quad \mathbf{B}^2 = \mathbf{K}.$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ defined in (166) and (168) may then be said to be "square roots" of the bounded positive definite matrices $\mathbf{E} - \mathbf{K}$ and $\mathbf{K}$ respectively. The matrices $\mathbf{A}$ and $\mathbf{B}$ in (166) and (168) are in addition positive definite but inasmuch as we shall not need this fact in the following we give no demonstration. The matrices $\mathbf{A}$ and $\mathbf{B}$ are obviously Hermitian and commutable, i. e.

$$(170) \quad \mathbf{AB} = \mathbf{BA} = \mathbf{A}^* \mathbf{B}^* = \mathbf{B}^* \mathbf{A}^* \quad \text{where} \quad \mathbf{A}^* = \bar{\mathbf{A}}', \mathbf{B}^* = \bar{\mathbf{B}}'.$$

We now employ the results of the previous paragraph in the demonstration † of the

LEMMA. If K is any bounded Hermitian matrix whose compound form $\Phi[K; x]$ is bounded by the inequalities (164), i. e.

$$(171) \quad 0 < \theta_1 < \Phi[K; x] < \theta_2 < 1 \quad \text{for } |x| = 1,$$

then the matrix polynomials

$$(172) \quad \pi_{n,m}(K) \quad \text{where} \quad \pi_{n,m}(x) = x^m(1-x)^{n-m}; \\ (m = 0, 1, \dots, n); (n = 0, 1, \dots),$$

are not-negative definite matrices, i. e.

$$(173) \quad 0 \leq \Phi[\pi_{m,n}(K); x] \quad \text{for } |x| = 1; \\ (m = 0, 1, \dots, n); (n = 0, 1, \dots).$$

In order to demonstrate (173) we employ the "square root" representation (167), (169) for the matrices $E - K$ and K respectively in the matrix polynomials (172) to obtain with the aid of the multiplication theorems (Faltungsätze) of Hilbert and the commutability property (170)

$$\pi_{m,n}(K) = (B^2)^m(A^2)^{n-m} = (B^m A^{n-m})(B^m A^{n-m}) = (B^m A^{n-m})(B^m A^{n-m})^*; \\ (m = 0, 1, \dots, n; n = 0, 1, \dots),$$

from which (173) follows at once inasmuch as the Hermitian matrix $(B^m A^{n-m})(B^m A^{n-m})^*$ is known to be non-negative definite.

We now consider the problem of moments ‡ for a finite interval, viz.

$$(174) \quad c_l = \int_0^1 \mu^l d\rho(\mu); \quad (l = 0, 1, \dots),$$

† The demonstration of this lemma may also be seen as a consequence of a theorem given by F. Riesz in "R", p. 129, which states that for a Hermitian matrix H whose compound form satisfies the inequalities

$$m \leq \Phi[H; x] \leq M \quad \text{for } |x| = 1,$$

the compound form of the matrix polynomial

$$f(H) \quad \text{where} \quad f(x) = \sum_{\nu=0}^n c_\nu x^\nu,$$

constructed from the matrix H , satisfies the inequalities

$$\lambda \leq \Phi[f(H); x] \leq \Lambda \quad \text{for } |x| = 1,$$

where

$$\lambda \leq f(\mu) \leq \Lambda \quad \text{for } m \leq \mu \leq M.$$

In the demonstration for this theorem, given by Riesz, use is made, however, of the truncated matrices of the matrix $f(H)$ whereas in the direct demonstration of the lemma given above, we avoid this procedure.

‡ See "Ha", pp. 222-232.

in which the c_l are preassigned real numbers and there is desired a real monotone non-decreasing function $\rho(\mu)$ of the real argument μ which satisfies the system (174) of Stieltjes integral equations. According to Hausdorff ‡ this problem possesses a solution then and only then if the sequence $\{c_l\}$ is completely monotone, i. e. if and only if the following inequalities hold:

$$(175) \quad \Delta^{n-m} c_m \geq 0; \quad (m = 0, 1, \dots, n); \quad (n = 0, 1, \dots),$$

where

$$\Delta c_m = c_m - c_{m+1}, \quad \Delta^j = \Delta^{j-1} \Delta; \quad (j = 2, 3, \dots),$$

so that

$$\Delta^{n-m} c_m = c_m - \binom{n-m}{1} c_{m+1} + \dots + (-1)^{n-m} c_n; \\ (m = 0, 1, \dots, n); \quad (n = 0, 1, \dots).$$

As is well known one may then insure the uniqueness of the solution $\rho(\mu)$ of (174) at all points of its interval of definition, the closed interval $[0, 1]$, by prescribing, for instance, the normalizing conditions

$$(176) \quad 0 = \rho(0) \quad \text{and} \quad \rho(\mu - 0) = \rho(\mu), \quad 0 < \mu < 1,$$

for the solution $\rho(\mu)$ of (174), obtaining as a consequence of the assumption $\rho(0) = 0$ from (174) for $l = 0$

$$(177) \quad \rho(1) - \rho(0) = \rho(1) = c_0.$$

For our purposes we consider the problem of moments arising from (174) if one places

$$(178) \quad c_l = \Phi[K^l; x], \quad |x| = 1, \quad K^0 = E; \quad (l = 0, 1, \dots),$$

in which x is an arbitrary but fixed unit vector and K a Hermitian matrix whose compound form $\Phi[K; x]$ is bounded by the inequalities (164), i. e. (171). We seek a monotone non-decreasing function $\rho(\mu; x)$ of the real argument μ and the components $x_1, x_2, \dots$ of the above given unit vector x which satisfies the system (174), (178) of Stieltjes integral equations, i. e. is a solution of the problem of moments

$$(179) \quad \Phi[K^l; x] = \int_0^1 \mu^l d\rho(\mu; x), \quad |x| = 1; \quad (l = 0, 1, \dots).$$

That the moment problem (179) possesses, for any given unit vector x , such

‡ See "Ha", p. 226. Cf. also I. J. Schoenberg, "On Finite and Infinite Completely Monotonic Sequences," *Bulletin of the American Mathematical Society*, Vol. 38, No. 2 (1932), pp. 72-76.

a monotone solution follows from the above lemma inasmuch as the Hausdorff conditions (175) are, for the moment problem (179)

$$\Delta^{n-m}\Phi[K^m; x] \geq 0, \quad \text{where} \quad \Delta^{n-m}\Phi[K^m; x] \\ \equiv \Phi[(K^m - \binom{n-m}{1}K^{m+1} + \dots + (-1)^{n-m}K^n); x] = \Phi[K^m(E - K)^{n-m}; x],$$

and which are, from (172) and (173), obviously fulfilled. In place of the normalizing conditions (176) prescribed for the solution $\rho(\mu)$ of (174) we now prescribe

$$(180) \quad 0 \equiv \rho(0; x) \quad \text{and} \quad \rho(\mu - 0; x) \equiv \rho(\mu; x) \quad \text{for} \quad 0 < \mu < 1; \quad |x| = 1,$$

for the solution $\rho(\mu; x)$ of (179) obtaining in place of (177)

$$(181) \quad \rho(1; x) - \rho(0; x) \equiv \rho(1; x) \equiv 1 \quad \text{for} \quad |x| = 1,$$

the solution $\rho(\mu; x)$ of (178) being thereby uniquely determined, for any given unit vector x , at all points of the closed interval $[0, 1]$.

We now define the function $\rho(\mu; x)$ also for $-\infty < \mu < 0$ and $1 < \mu < +\infty$ taking it identically equal to 0 and $+1$ in the two ranges respectively. With this understanding we write in place of (178)

$$(182) \quad \Phi[K^l; x] = \int_{-\infty}^{+\infty} \mu^l d\rho(\mu; x); \quad (l = 0, 1, \dots).$$

Let H be an arbitrary bounded Hermitian matrix. Since H is bounded there exist two real numbers α ($\alpha > 0$) and β so that the matrix

$$(183) \quad K = \alpha H + \beta E$$

fulfills the condition (164). In denoting by $\rho(\mu; x)$ the solution of the moment problem (179), (180), (181) we introduce a function $\sigma(\mu; x)$ by means of the definition

$$(184) \quad \sigma(\mu; x) \equiv \rho(\alpha\mu + \beta; x), \quad \alpha > 0, \quad -\infty < \mu < +\infty.$$

From (182), (183) and (184) we then have

$$\Phi[H^l; x] = \int_{-\infty}^{+\infty} \mu^l d\sigma(\mu; x),$$

i. e. the spectral representation of Hilbert.

VITA.

I, Monroe H. Martin, was born at Lancaster, Pennsylvania, on February 7, 1907, the first son of Amos and Mary Martin. I received my early education in the public schools of Annville, Pennsylvania, and proceeded to Lebanon Valley College, Annville (1924-28), where I obtained the degree of B.S. I came to the Johns Hopkins University in the autumn of 1928, where I was granted an instructorship in mathematics, and have studied up to the present time.

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